Commentary by me on Following Paper Brian Scannell 30/9/01

i) There could be a problem with this in the fact that in Section 10 for rational values of $\cos^n\theta+\sin^n\theta$ as a summation of $\cos n\theta$, it is wrong to say that if each individual term of the expansion reproduced below

$$2^{n-2}(\cos^n\theta+\sin^n\theta)=\cos n\theta+\frac{n(n-1)}{2!}\cos(n-4)\theta$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}\cos(n-8)\theta+...$$
$$...+1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}\text{..... n even, n/2 even}$$

is irrational then the summation is irrational. Since summation of irrational numbers is not closed i.e. $\alpha+\beta\neq\chi$ where α,β and χ are all irrational i.e. (2+$\sqrt{2}$) +(3-$\sqrt{2}$)=5 which isn't irrational.
Refutation: while this is in general true for composite irrational numbers a+b where at least one of a or b are irrational this is not true for single irrational numbers *I* (excepting I +-I=0 if 0 is irrational.)
As is the case here (?)
ii) Thing to prove;
$x^n+y^n=z^n$, if x,y,z integer, for n>2 then $R(\theta)=z/(\cos^n\theta+\sin^n\theta)^{1/n}$ is irrational.
For n=2 $R(\theta)$ can = z an integer and then x,y, $R(\theta)$ (=z) is a Pythagorean Triple. For n>2, since $R(\theta)$ is irrational, if there is an integer solution to $x^n+y^n=z^n$ then z= integer and therefore $(\cos^n\theta+\sin^n\theta)^{1/n}$ is irrational but $R(\theta)^2$ is rational i.e. $(\cos^n\theta+\sin^n\theta)^{2/n}$ is rational . If $(\cos^n\theta+\sin^n\theta)^{1/n}$ is rational then z must be irrational and therefore there is no integer solution to $x^n+y^n=z^n$.
iii) Whilst it can be proved that for all theta (except 60 deg) rational then cos theta is irrational. Is the reverse true. for all theta irrational is cos theta rational (except 60 degrees). Even if it is true it may not be an entire set. Can cos theta rational yield theta that is irrational as well as rational?

Discussion on Fermat's Last Theorem using Mathematics Contemporary to Fermat

by
Brian Scannell

17/6/01

The following discussion is for rational theta only.
Cosines and sines of irrational theta are rational, in particular, for Pythagorean triples the sine and cosine of irrational theta are both rational. For other irrational theta the sine is rational but the cosine is irrational or vice versa.

Introduction

Fermat's Last Theorem (FLT) states that there are no positive integers x,y,z and n for n>2 such that,

$$x^n + y^n = z^n . \qquad (1)$$

In 1994 Professor Wiles solved Fermat's Last Theorem but added:

> *"Fermat couldn't possibly have had this proof. It's a 20th-century proof. There's no way this could have been done before the 20th-century." (see Ref.1)*

Fermat formulated his theorem between 1630 to 1654 (maybe 1637) in annotating his copy of Bachet's translation of Diophantus' Arithmetica with (translated into English and using modern terminology),

> *There are no positive integers such that $x^n + y^n = z^n$, for n>2. I've found a remarkable proof of this fact, but there is not enough space in the margin to write it.*

His proof has never been found. But if he had solved it he would have only be able to use the mathematics available at the time. Trigonometric functions were available at this time (see Appendix 1).

The following discussion uses trigonometric functions. For ease of visualisation it also uses graphical outputs.

If Fermat could prove the irrationality of the trigonometric expressions given in Note 5 he could have been able to prove his last theorem. Without the graphs the summary proof presented here wouldn't be much bigger than Bachet's margin (maybe!).

Summary Proof.

1) Note 1 depicts graphically the form of $x^n + y^n = z^n$ for $x, y \text{ and } z > 0$. I have called these the Fermat Charts.

2) A plan view of the Fermat Charts show the integer contours of z for a given n for the solution of $x^n + y^n = z^n$. I have called these Fermat Contour Plots and are shown in Note 2.

3) Note 3 constructs a line on the Fermat Contour Plots that starts from the origin and ends on a integer z contour line. I have called this a Fermat Vector $R(\theta)$ and defined over the range $0° < \theta < 90°$.

4) The crux of this whole proof is the step given in Note 4.
From observations on the Fermat Vector, algebraic manipulation shows $R(\theta)$ is given by

$$R(\theta) = z/(\cos^n \theta + \sin^n \theta)^{1/n}$$

Plots show that for $n > 2$ $R(\theta)$ is maximum for $\theta = 45°$ and is symmetric about that angle.

For n=2 $R(\theta)$ is constant.

The normalised Fermat Vector $\hat{R}(\theta)$ given by $\frac{R(\theta)}{z}$

For there to be a solution to Fermat's equation (1) as shown in the Fermat Contour Plots, the vector $R(\theta)$ must finish on a integer contour *and* have components x and y which are also integer.

Note 4 shows this leads to the re-statement that if FLT is true then,

$$(\cos^n \theta + \sin^n \theta)^{2/n} \text{ is rational for n} = 2$$

and

$$(\cos^n \theta + \sin^n \theta)^{2/n} \text{ is irrational for n} > 2$$

where, according to the Fermat Contour Plots, θ is defined over the range $0° < \theta < 90°$.

5) Note 5 discusses the rational roots of polynomial equations and shows that a number of the form $\sqrt[n]{a}$, where a and n are positive integers, is either irrational or integer; in the latter case a is the nth power of an integer.

6) But by looking at the Fermat Vector in the Fermat Plot,

$$x^2 + y^2 = R(\theta)^2.$$

Since x and y are integers $R(\theta)^2$ is an integer and therefore from Note 5 $R(\theta)$ is irrational unless $R(\theta)$ is an integer, and in this case is x,y, $R(\theta)$ is a Pythagorean Triple (PT) and $R(\theta) = z/(\cos^n\theta + \sin^n\theta)^{1/n} = z \therefore n = 2$. Therefore for n>2 if there is an integer solution to $x^n + y^n = z^n$, $R(\theta)$ has to be irrational and $R(\theta)^2$ has to be an integer.

Additionally since $\sin(\theta) = y/R(\theta)$ and $\cos(\theta) = x/R(\theta)$ for there to be an integer solution to $x^n + y^n = z^n$, $\sin(\theta)$ and $\cos(\theta)$ must be irrational since $R(\theta)$ is irrational.

Note 6 discusses trigonometric numbers and shows that for rational θ, $\cos\theta$ and $\sin\theta$ are irrational apart from the values 0, ±0.5 and ±1 i.e. over the range $0° < \theta < 90°$ $\cos\theta$ is only rational for $\theta = 60°$ and $\sin\theta$ is only rational for $\theta = 30°$.

Therefore for there to be an integer solution to $x^n + y^n = z^n$, θ must be rational in order to make $R(\theta)$ irrational (apart from $\theta = 60°$ for $\cos\theta$ and $\theta = 30°$ for $\sin\theta$.)

Statement 1
Since $R(\theta) = z/(\cos^n\theta + \sin^n\theta)^{1/n}$ *and* $R(\theta)^2 = z^2/(\cos^n\theta + \sin^n\theta)^{2/n}$ *for there to be an positive integer solution to* $x^n + y^n = z^n$ *for n>2,* $(\cos^n\theta + \sin^n\theta)^{1/n}$ *has to be irrational and* $(\cos^n\theta + \sin^n\theta)^{2/n}$ *has to be rational.*

(Rational θ*?)*

Notes 7 to 10 described in steps 7 to 10 below show that for all rational θ, $(\cos^n\theta + \sin^n\theta)^{1/n}$ is indeed irrational but $(\cos^n\theta + \sin^n\theta)^{2/n}$ is also irrational.
This is contrary to Statement 1 and therefore there are no positive integer solution to $x^n + y^n = z^n$ for n>2.

The steps needed to show $(\cos^n\theta + \sin^n\theta)^{1/n}$ and $(\cos^n\theta + \sin^n\theta)^{2/n}$ are irrational for rational θ are given below in 7 to 10.

7)The first step is to find the rational values of $\cos^n\theta + \sin^n\theta$. This is achieved in Notes 7 to 10.
The steps are to expand $\cos^n\theta$ in terms of $n\theta$ as in Note 7 and $\sin^n\theta$ in terms of $n\theta$ as in Note 8.
Note 9 then gives the combined expansion of $\cos^n\theta + \sin^n\theta$ in terms of $n\theta$.

Note 10 finds that for rational theta the only rational values of $\cos^n\theta + \sin^n\theta$ over the angular range $0 < \theta < 90°$ occur for n even and
$\theta = 15°, 22.5°, 30°, 45°, 60°, 67.5°$ and $75°$.

8) Note 11 finds the following rational values of $(\cos^n\theta + \sin^n\theta)^2$,
for n even, theta rational,

$(\cos^n\theta+\sin^n\theta)^2$ is rational for $\theta=15,22.5,30,45,60,67.5\text{ and }75°$,
for n odd,
$(\cos^n\theta+\sin^n\theta)^2$ is rational for $\theta=15,45\text{ and }75°$.
For all other θ, $(\cos^n\theta+\sin^n\theta)^2$ is irrational over the angular range $0°<\theta<90°$.

9) Note 12 shows that if α is irrational then $\alpha^{1/n}$ is irrational.
Letting $\alpha=(\cos^n\theta+\sin^n\theta)^2$ for all rational θ and n except those in 7) above, this shows $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational.

10) The irrationality of $(\cos^n\theta+\sin^n\theta)^{2/n}$ for the few remaining rational θ given in 7) is shown on a case by case basis in Note 13.

11) As a summary $(\cos^n\theta+\sin^n\theta)^{2/n}$ has been shown to be irrational for rational θ and n>2.

This is the same as saying there are no integer x,y,z for integer n>2 for which

$x^n+y^n=z^n$.

Also $(\cos^n\theta+\sin^n\theta)^{2/n}$ is rational for n=2 from the identity $\cos^2\theta+\sin^2\theta=1$ therefore there are solutions for integer x,y,z of

$x^n+y^n=z^n$ for n=2

i.e.
$x^2+y^2=z^2$ has integer solutions.

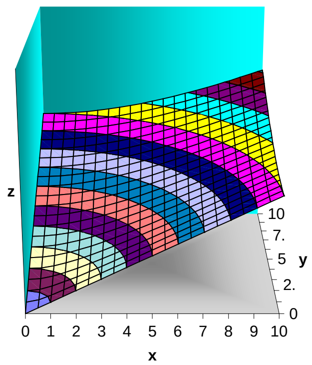
For rational theta, showing the irrationality of $(\cos^n\theta+\sin^n\theta)^{2/n}$ for n>2 thus shows Fermat's Last Theorem is true for rational theta.

Note 1

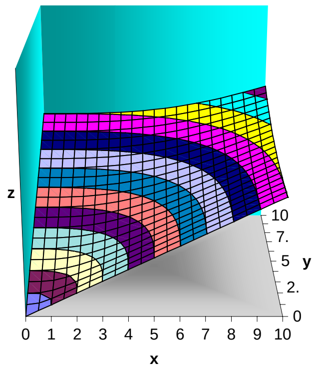
Plots of $x^n + y^n = z^n$ i.e. z=f(x,y)= $(x^n + y^n)^{1/n}$ for positive real z (called Fermat Charts) are shown here for $0 \le x, y \le 10$ and n=2,3 and 6.

For ease of visualisation, contours are shown for integer values of z

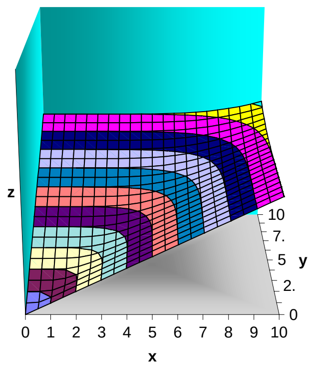
Fermat Chart for n=2



Fermat Chart for n=3

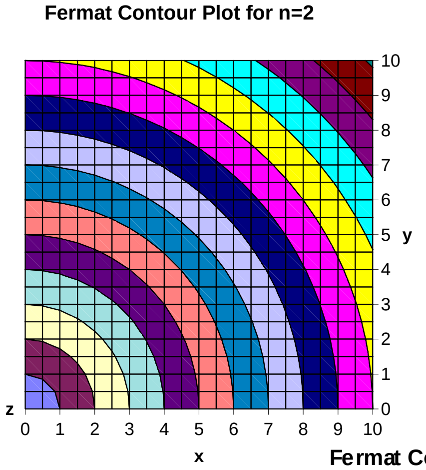


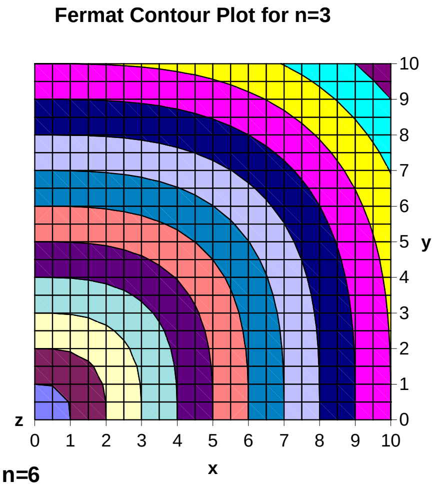
Fermat Chart for n=6

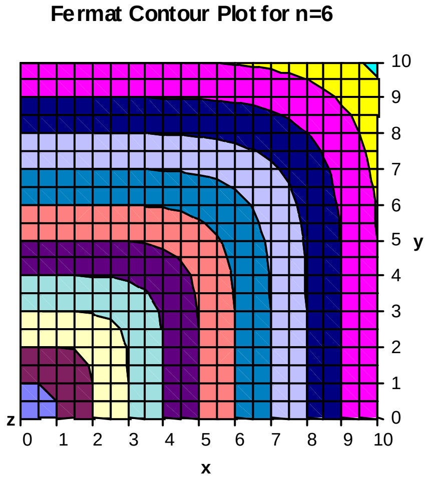


Note 2

Plan views of the Fermat Charts (called Fermat Contour Plots) are shown here for the above three cases. Again integer contours of z are shown.

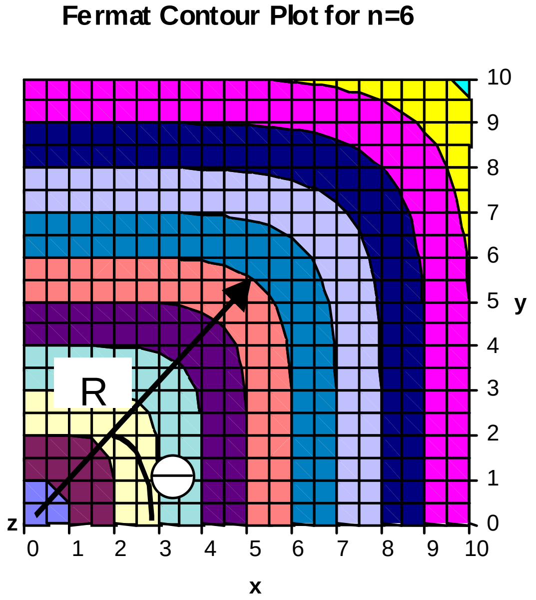






Note 3

Construct a radius on the Fermat Contour Plot. The radius is drawn from the origin and ends on a contour line. This is called the Fermat Vector, R(θ).



Note 4
Form of R(θ).

From trigonometry,

$R^2 = x^2 + y^2$ 4.1

and

$$\left.\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \end{aligned}\right\}$$ 4.2

Now

$x^n + y^n = z^n$..4.3

Inserting equation 4.2 into 4.3 gives,

$R^n \cos^n\theta + R^n \sin^n\theta = z^n$.

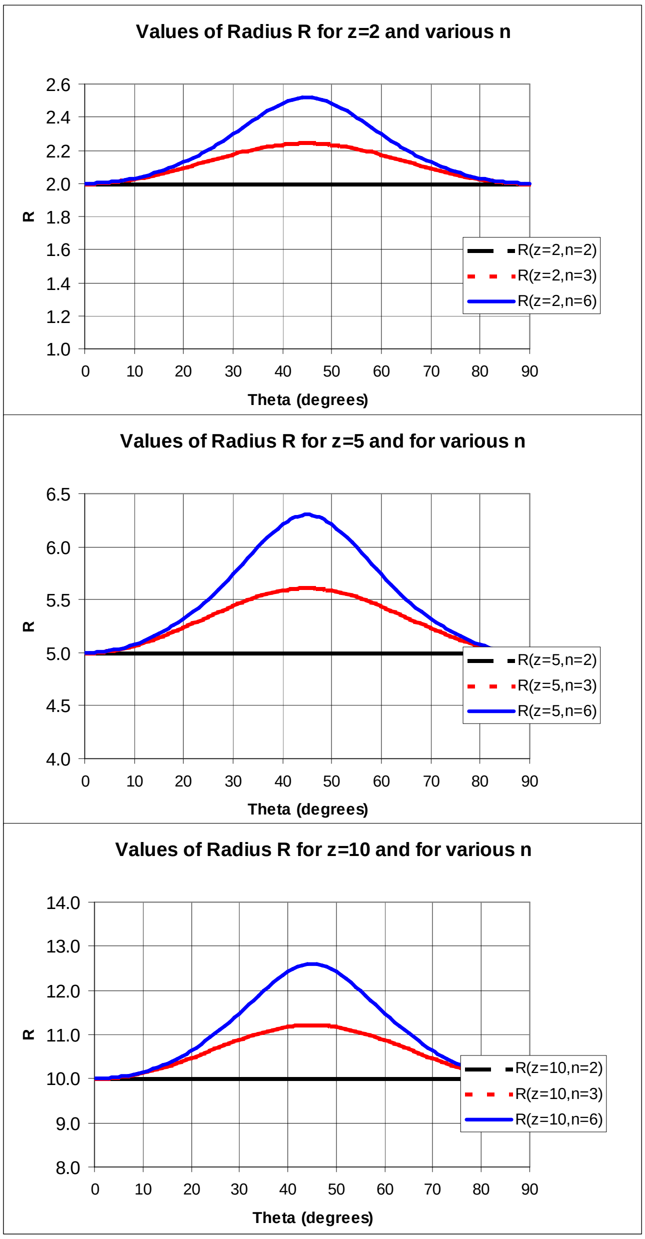
$\Rightarrow R^n(\cos^n\theta + \sin^n\theta) = z^n$

$\Rightarrow R = z/(\cos^n\theta + \sin^n\theta)^{1/n} = \mathrm{R}(\theta)$.................4.4

In order that x or y not be zero, θ is defined over the range,

$0 < \theta < 90$.

4.1 Values of R(θ) are shown for z=2,5 and 10 for n=2,3,and 6.



Note:
4.1.1 R(θ) = constant = z for n=2.
this is shown by putting n=2 into equation 4.4

$$R(\theta) = z/(\cos^2\theta + \sin^2\theta)^{1/2}$$

$$\text{standard trig. identity gives } \cos^2\theta + \sin^2\theta = 1$$

$$\therefore$$

$$R(\theta) = z$$

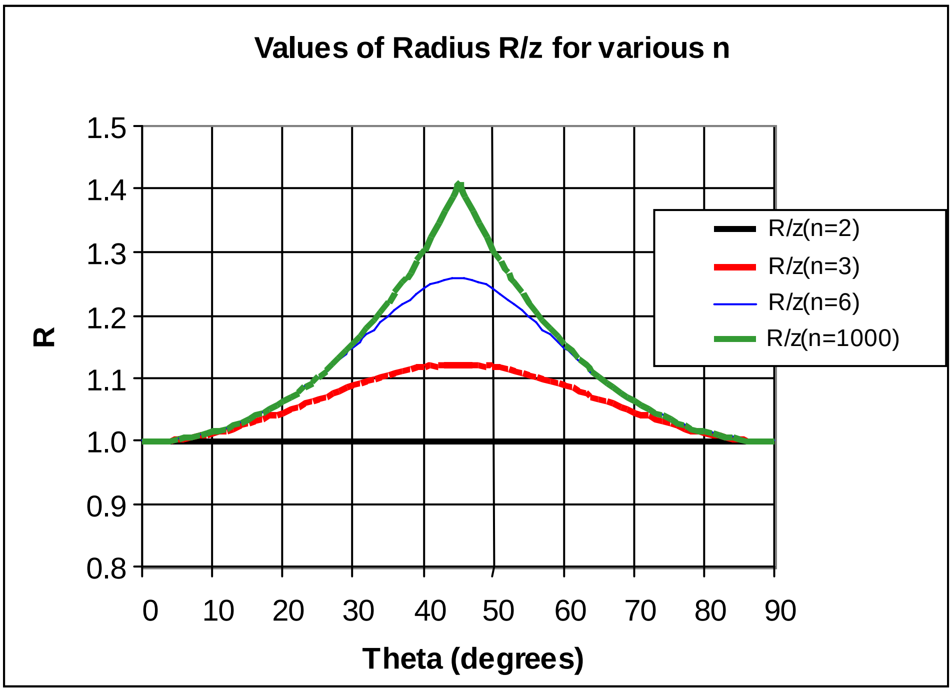
4.1.2 The maximum value of R(θ) occurs for θ = 45 degrees and R(θ) is symmetric about that angle.

4.2 R(θ) can be normalised to give a curve independent of z
From eqn. 4.4.

$R(\theta) = z/(\cos^n\theta + \sin^n\theta)^{1/n}$

$\Rightarrow R(\theta)/z = 1/(\cos^n\theta + \sin^n\theta)^{1/n}$ 4.5

Where $R(\theta)/z$ is the normalized Fermat Vector $\hat{R}(\theta)$ which has a value ranging from 1 to 1.42 for all n and θ.



Comment

(i) As $n \Rightarrow \infty$, $\hat{\mathrm{R}}(\theta)\max \Rightarrow \sqrt{2}$, as shown below.

$R(\theta)/z = 1/(\cos^n\theta + \sin^n\theta)^{1/n}$

at $\theta = 45$ degrees

$\cos\theta = \sin\theta = 1/\sqrt{2}$

$\therefore$

$R(\theta)/z = 1/[(1/\sqrt{2})^n + (1/\sqrt{2})^n]^{1/n}$

$= 1/[2(1/\sqrt{2})^n)^{1/n}$

$= 1/[2^{1/n}(1/\sqrt{2})]$

$= \sqrt{2}/2^{1/n}$

$\therefore$ as $n \Rightarrow \infty$

$R(\theta)/z = \hat{R}(\theta) \Rightarrow \sqrt{2}$

4.3 Observation. For there to be a solution to Fermat's equation (1) as shown in the Fermat Contour Plots, the Fermat Vector R(θ) must finish on a integer contour *and* be coincident with an intersection of integer grid points i.e. have components x and y which are also integer.

For this to be the case, combining equations 4.1 and 4.4 give,

$x^2 + y^2 = \mathrm{R}(\theta)^2 = z^2 /(\cos^n \theta + \sin^n \theta)^{2/n}$ 4.6

Suppose Fermat's Theorem of eqn (1) is false i.e. there are integer solutions x,y,z for integer n>2. Therefore since x and y are integers then,

x^2 and y^2 are integers.

Therefore,

$x^2 + y^2$ is an integer, call it A.

Therefore $R(\theta)^2$ is also the integer A.

Therefore

$z^2 /(\cos^n \theta + \sin^n \theta)^{2/n}$ is also the integer A.

Since z is an integer, z^2 is also an integer, call it B.

Therefore,

$A/B = \hat{R}(\theta)^2 = 1/(\cos^n \theta + \sin^n \theta)^{2/n}$ is a rational number.

i.e. the normalized Fermat Vector squared is rational.

Therefore,

$B/A = (\cos^n \theta + \sin^n \theta)^{2/n}$ is also rational and is equal to, $1/\hat{R}(\theta)^2$ the reciprocal squared normalized Fermat Vector.

4.4 Re-statement of Fermat's Last Theorem.

If FLT is false, then there are positive integers x,y,z and n>2 such that

$x^n + y^n = z^n$.

which is equivalent to the statement,

$(\cos^n \theta + \sin^n \theta)^{2/n}$ is a rational number.

In other words FLT is true if

$(\cos^n \theta + \sin^n \theta)^{2/n}$ is a rational number for n=2,
and

$(\cos^n \theta + \sin^n \theta)^{2/n}$ is irrational for n>2, where, as shown by the Fermat Contour Plots, $0° < \theta < 90°$.

Note 5 Rational Roots of Polynomial Equations

The following is taken from I.Niven Ref. 3.

Theorem 5.1
Let *u, v, w* be integers such that *u* is a divisor of *vw*, and *u* and *v* have no prime factors in common. Then *u* is a divisor of *w*. More generally, if *u* is a divisor of $v^n w$, where n is any positive integer and *u* and *v* have no prime factors in common, then *u* is a divisor of *w*.

Example
u = 4, *v* = 5, v^3w = 500.
4 and 5 have no prime factors in common and 4 divides 500. Also 4 divides $500/5^3$ = 4.

Proof

The main ingredient is the Fundamental Theorem of Arithmetic (see for example, Ref. 3 for a proof of this) which assures us that there is only one way to factor u, v, w into prime factors.

Theorem 5.2
Consider any polynomial equation with integer coefficients,

$$c_n x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_2 x^2 + c_1 x + c_0 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots 5.1$$

If this equation has a rational root a/b, where a/b is presumed to be in its lowest terms, then a is a divisor of c_0 and b is a divisor of c_n.

Proof
Substitute a/b in eqn 5.1 then multiply by b^n

$$c_n a^n + c_{n-1}a^{n-1}b + c_{n-2}a^{n-2}b^2 + ... + c_2 a^2 b^{n-2} + c_1 ab^{n-1} + c_0 b^n = 0 \quad \ldots\ldots 5.2$$

Which can be written as

$$c_n a^n = b(-c_{n-1}a^{n-1} - c_{n-2}a^{n-2}b - ... - c_2 a^2 b^{n-3} - c_1 ab^{n-2} - c_0 b^{n-1})$$

This shows b is a divisor of $c_n a^n$ and by applying Theorem 5.1 with *u, v, w* replaced by *b, a,* and c_n, we conclude that *b* is a divisor of c_n.

Eqn 5.2 can be written as,

$$c_0 b^n = a(-c_n a^{n-1} - c_{n-1}a^{n-2}b - ... - c_2 ab^{n-2} - c_1 b^{n-1})$$

This shows *a* is a divisor of $c_0 b^n$ and by applying Theorem 5.1 with *u*, *v*, and *w* replaced by *a*, *b*, and c_o, we conclude *a* is a divisor of c_0.

Corollary 1

Consider an equation of the form,

$$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + ... + c_2x^2 + c_1x + c_0 = 0 \text{ ....5.3}$$

with integer coefficients. If such an equation has a rational root, it is an integer, and this integer root is a divisor of c_0.

Proof

Consider any rational root *a/b* with *b* a positive integer. According to Theorem 5.1 *b* must be a divisor of c_n; that is *b* is a divisor of 1; that is *b* =1. Consequently any rational root is of the form *a/1*, so it is an integer *a*. Also by Theorem 5.1 *a* is a divisor of c_0.

Corollary 2

A number of the form $\sqrt[n]{a}$, where a and n are positive integers, is either irrational or integer; in the latter case a is the nth power of an integer.

Proof

This follows from Corollary 1 because $\sqrt[n]{a}$ is a root of $x^n - a = 0$, which is an equation of the form 5.3, and if this equation has a rational root it must be an integer. Furthermore , if $\sqrt[n]{a}$ is an integer, say k, then $a = k^n$, since $\sqrt[n]{k^n} = k$.

Note 6
Trigonometric Numbers
This proof is adapted from I. Niven Ref 3
Theorem 6.1
Let θ = 180k° be an angle whose measurement in degrees is a rational number, that is, k is rational. Then cos θ and sin θ are irrational apart from the values 0, ±0.5 and ±1.

In order to prove this we need to use the trigonometric expansions of functions of multiple angles in a series of descending powers. This is given in eqn 6.8 below.

From the trigonometric identity,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\textit{putting}\, \theta = A,\ \theta = B$$
$$\text{we get}$$
$$\cos 2\theta = (\cos\theta)^2 - (\sin\theta)^2 \qquad \text{..........6.1}$$

using,

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\text{putting}\, \theta = A,\ \theta = B \text{ this becomes,}$$
$$1 = (\cos\theta)^2 + (\sin\theta)^2 \qquad \text{.......6.2}$$

substituting for $(\sin\theta)^2$ from eqn 6.2 into eqn 6.1 gives,

$$\cos 2\theta = 2(\cos\theta)^2 - 1$$

Multiplying by 2 gives,

$$\boxed{2\cos 2\theta = (2\cos\theta)^2 - 2} \qquad \text{....6.3}$$

We can continue by putting

$$\theta = 2A,\ \theta = B \text{ in}$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\text{giving}$$
$$\cos 3\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta \qquad \text{................6.4}$$

and

$\theta = 2A,\ \theta = B$ in

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

giving

$$\cos\theta = \cos 2\theta \cos\theta + \sin 2\theta \sin\theta \qquad .... 6.5$$

Adding eqn 6.4 to 6.5 gives

$$\cos 3\theta + \cos\theta = 2\cos 2\theta \cos\theta$$

Substituting for $2\cos 2\theta$ from eqn 6.3

$$\cos 3\theta = (2\cos\theta)^2 \cos\theta - 3\cos\theta$$

Multiplying by 2 gives

$$\boxed{2\cos 3\theta = (2\cos\theta)^3 - 3(2\cos\theta)}$$

A recursive formula for this is

$$\boxed{2\cos(n+1)\theta = (2\cos n\theta)(2\cos\theta) - 2\cos(n-1)\theta}$$

The expansions continue,

$$2\cos 4\theta = (2\cos\theta)^4 - 4(2\cos\theta)^2 + 2$$
$$2\cos 5\theta = (2\cos\theta)^5 - 5(2\cos\theta)^3 + 5(2\cos\theta)$$
$$2\cos 6\theta = (2\cos\theta)^6 - 6(2\cos\theta)^4 + 9(2\cos\theta)^2 - 2$$
$$2\cos 7\theta = (2\cos\theta)^7 - 7(2\cos\theta)^5 + 14(2\cos\theta)^3 - 7(2\cos\theta)$$
$$2\cos 8\theta = (2\cos\theta)^8 - 8(2\cos\theta)^6 + 20(2\cos\theta)^4 - 16(2\cos\theta)^2 + 2$$
$$2\cos 9\theta = (2\cos\theta)^9 - 9(2\cos\theta)^7 + 27(2\cos\theta)^5 - 30(2\cos\theta)^3 + 9(2\cos\theta)$$

Ref 3 gives the general trigonometric expansion

$$\cos n\theta = 2^{n-1}\cos^n\theta - \frac{n}{1!}2^{n-3}\cos^{n-2}\theta + \frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}\theta - ....6.6$$

where the general term is

$$(-1)^r \frac{n(n-r-1)...(n-2r+1)}{r!} 2^{n-2r-1}\cos^{n-2r}\theta \qquad \text{for n + ve integer.}$$

Multiplying eqn. 6.6 by 2 we get,

$$(2\cos n\theta) = (2\cos\theta)^n - \frac{n}{1!}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \text{.............}6.7$$

where the general term is

$$(-1)^r \frac{n(n-r-1)...(n-2r+1)}{r!}(2\cos\theta)^{n-2r} \quad .$$

for example, the coefficient of $(2\cos\theta)^0$ for n=6 is,

$r = 3$ for $2n - r = 0$

$\therefore$ in the general term of eqn 6.1 above,

$n - r - 1 = 6 - 3 - 1 = 2$

$n - 2r + 1 = 6 - 6 + 1 = 1$

$\therefore$ last coefficient of $(2\cos 6\theta)$ is,

$$(-1)^3 \frac{.6.2.1}{3!} = -2$$

$$(2\cos n\theta) = (2\cos\theta)^n + c_{n-2}(2\cos\theta)^{n-2} + c_{n-4}(2\cos\theta)^{n-4} + \text{.............}6.8$$
$$... + c_{n-2r}(2\cos\theta)^{n-2r},$$

where the coefficients c_k are integers.

More generally this can be written

$$(2\cos n\theta) = (2\cos\theta)^n + c_{n-1}(2\cos\theta)^{n-1} + c_{n-2}(2\cos\theta)^{n-2} + \text{.............}6.9$$
$$... + c_2(2\cos\theta)^2 + c_1(2\cos\theta) + c_0$$

where the coefficients c_k are integers

and $c_1, c_3, c_5 ...$ are zero for $n = \text{even}$

$c_0, c_2, c_4 ...$ are zero for $n = \text{odd}$.

Substitute 180k° for θ in $2\cos n\theta$

$\therefore 2\cos n\theta = 2\cos 180nk = 2\cos 180m = \pm 2$

where $m = \text{integer}$ since $n = \text{integer}$ and k rational.

$$\therefore (2\cos k180)^n + c_{n-1}(2\cos k180)^{n-1} + c_{n-2}(2\cos k180)^{n-2} +$$
$$... + c_2(2\cos k180)^2 + c_1(2\cos k180) + c_0 \pm 2 = 0$$

where the coefficients c_k are integers

This an equation of the form of eqn 5.3 where $x = 2\cos k180$ is a root of this equation and Corollary 1 says that if $2\cos k180$ is rational then it is an integer.

$\therefore 2\cos k180 = 0, \pm 1, \pm 2, \pm 3 ...$

But the maximum value that $2\cos k180$ can be is ± 2 for $k = 0,1,2....$

$\therefore 2\cos k180 = 0, \pm 1$ or ± 2

$\Rightarrow \cos k180 = 0$, ±0.5 or ±1

Since $\sin k180 = \cos(90 - k180) = \cos((1/2 - k)180) = \cos(j180)$ where j is rational. It follows $\sin(k180)$ has the same rational values as $\cos(k180)$ and Theorem 6.1 is proved.

Therefore for rational θ, $\cos\theta$ and $\sin\theta$ are irrational apart from the values 0, ±0.5 and ±1.

Table 6.1 Rational Values of $\cos\theta$

$\cos\theta$	θ examples	θ general
0	90,270,450,630,810	180k+90° k=0,1,2,3…
+0.5	60,300,420,660,780	360k±60° k=0,1,2,3…
-0.5	120,240,480,600,840	180k±60° k=1,3,5,7…
+1	0,360,720,1080,1440	360k° k=0,1,2,3…
-1	180,540,900,1260,1620	180k° k=1,3,5,7…

Table 6.2 Rational Values of $\sin\theta$

$\sin\theta$	θ examples	θ general
0	0,180,360,540,720	180k° k=0,1,2,3…
+0.5	30,150,390,510,750	360k+(90±60)° k=0,1,2,3…
-0.5	210,330,570,690,930	180k+(90±60)° k=1,3,5,7…
+1	90,450,810,1170,1530	360k+90° k=0,1,2,3…
-1	270,630,990,1350,1710	360k+270° k=0,1,2,3…

Note 7
Powers of $\cos\theta$ in terms of multiple angles

7.1 $\cos^n\theta$ for n even.

$$n = 2$$
$$2\cos^2\theta = \cos 2\theta + 1$$

$$n = 4$$
$$2^3\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$

$$n = 6$$
$$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$$

$$n = 8$$
$$2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$$

$$n = 10$$
$$2^9\cos^{10}\theta = \cos 10\theta + 10\cos 8\theta + 45\cos 6\theta + 120\cos 4\theta + 210\cos 2\theta + 126$$

$$n = 12$$
$$2^{11}\cos^{12}\theta = \cos 12\theta + 12\cos 10\theta + 66\cos 8\theta + 220\cos 6\theta + 495\cos 4\theta + 792\cos 2\theta + 462$$

In general

$$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots$$
$$\ldots + 1/2\,\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)} \text{ ..... n even} \qquad \text{....7.1}$$

7.2 $\cos^n\theta$ for n odd

$$n = 3$$
$$2^2\cos^3\theta = \cos 3\theta + 3\cos\theta$$

$$n = 5$$
$$2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$$

$$n = 7$$
$$2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$$

$$n = 9$$
$$2^8\cos^9\theta = \cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta$$

$n = 11$

$$2^{10}\cos^{11}\theta = \cos 11\theta + 11\cos 9\theta + 55\cos 7\theta + 165\cos 5\theta + 330\cos 3\theta + 462\cos\theta$$

In general:

$$2^{n-1}\cos^{n}\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots$$
$$\ldots + \frac{n!}{(\frac{n-1}{2}!)(\frac{n+1}{2}!)}\cos\theta \ldots\ldots \text{ n odd} \qquad \ldots\ldots\ldots..7.2$$

Note 8 Powers of $\sin\theta$ in terms of multiple angles

8.1 $\sin^n\theta$ for n even.

$n=2$

$2\sin^2\theta = -\cos 2\theta + 1$

$n=4$

$2^3\sin^4\theta = \cos 4\theta - 4\cos 2\theta + 3$

$n=6$

$2^5\sin^6\theta = -\cos 6\theta + 6\cos 4\theta - 15\cos 2\theta + 10$

$n=8$

$2^7\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$

$n=10$

$2^9\sin^{10}\theta = -\cos 10\theta + 10\cos 8\theta - 45\cos 6\theta + 120\cos 4\theta - 210\cos 2\theta + 126$

$n=12$

$2^{11}\sin^{12}\theta = \cos 12\theta - 12\cos 10\theta + 66\cos 8\theta - 220\cos 6\theta + 495\cos 4\theta - 792\cos 2\theta + 462$

In general

$$(-1)^{n/2}2^{n-1}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta - \ldots$$
$$\ldots + (-1)^{n/2}1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}\ldots\ldots \text{ n even} \quad \ldots\ldots\ldots 8.1$$

8.2 $\sin^n\theta$ for n odd.

$n=3$

$2^2\sin^3\theta = -\sin 3\theta + 3\sin\theta$

$n=5$

$2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$n=7$

$2^6\sin^7\theta = -\sin 7\theta + 7\sin 5\theta - 21\sin 3\theta + 35\sin\theta$

$n=9$

$2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$

$n=11$

$2^{10}\sin^{11}\theta = -\sin 11\theta + 11\sin 9\theta - 55\sin 7\theta + 165\sin 5\theta - 330\sin 3\theta + 462\sin\theta$

In general:

$$(-1)^{(n-1)/2}2^{n-1}\sin^n\theta = \sin n\theta - n\sin(n-2)\theta + \frac{n(n-1)}{2!}\sin(n-4)\theta - \ldots$$
$$\ldots + (-1)^{(n-1)/2}\frac{n!}{(\frac{n-1}{2}!)(\frac{n+1}{2}!)}\sin\theta\ldots\ldots \text{ n odd} \qquad \ldots.8.2$$

Note 9 $\cos^n\theta + \sin^n\theta$

9.1 $\cos^n\theta + \sin^n\theta$ **for n even, n/2 even.**

n=4

$2^2(\cos^4\theta + \sin^4\theta) = \cos 4\theta + 3$

n=8

$2^6(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$

n=12

$2^{10}(\cos^{12}\theta + \sin^{12}\theta) = \cos 12\theta + 66\cos 8\theta + 495\cos 4\theta + 462$

In general:

$$2^{n-2}(\cos^n\theta + \sin^n\theta) = \cos n\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos(n-8)\theta + \ldots + 1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)} \ldots\ldots \text{ n even, n/2 even} \qquad \ldots\ldots 9.1$$

9.2 $\cos^n\theta + \sin^n\theta$ **for n even, n/2 odd.**

n=2

$(\cos^2\theta + \sin^2\theta) = 1$

n=6

$2^4(\cos^6\theta + \sin^6\theta) = 6\cos 4\theta + 10$

n=10

$2^8(\cos^{10}\theta + \sin^{10}\theta) = 10\cos 8\theta + 120\cos 4\theta + 126$

In general:

$$2^{n-2}(\cos^n\theta + \sin^n\theta) = n\cos(n-2)\theta + \frac{n(n-1)(n-2)}{3!}\cos(n-6)\theta + \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos(n-10)\theta + \ldots + 1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)} \ldots\ldots \text{ n even, n/2 odd} \qquad \ldots\ldots 9.2$$

9.3 $\cos^n\theta + \sin^n\theta$ **for n odd.**

n=3

$$2^2(\cos^3\theta+\sin^3\theta)=(\cos 3\theta-\sin 3\theta)+3(\cos\theta+\sin\theta)$$

n=5

$$2^4(\cos^5\theta+\sin^5\theta)=(\cos 5\theta+\sin 5\theta)+5(\cos 3\theta-\sin 3\theta)+10(\cos\theta+\sin\theta)$$

n=7

$$2^6(\cos^7\theta+\sin^7\theta)=(\cos 7\theta-\sin 7\theta)+7(\cos 5\theta+\sin 5\theta)+21(\cos 3\theta-\sin 3\theta)+35(\cos\theta+\sin\theta)$$

n=9

$$2^8(\cos^9\theta+\sin^9\theta)=(\cos 9\theta+\sin 9\theta)+9(\cos 7\theta-\sin 7\theta)+36(\cos 5\theta+\sin 5\theta)$$
$$+84(\cos 3\theta-\sin 3\theta)+126(\cos\theta+\sin\theta)$$

n=11

$$2^{10}(\cos^{11}\theta+\sin^{11}\theta)=(\cos 11\theta-\sin 11\theta)+11(\cos 9\theta+\sin 9\theta)+55(\cos 7\theta-\sin 7\theta)$$
$$+165(\cos 5\theta+\sin 5\theta)+330(\cos 3\theta-\sin 3\theta)+462(\cos\theta+\sin\theta)$$

In general:

$$\begin{aligned}2^{n-1}(\cos^n\theta+\sin^n\theta)&=(\cos n\theta+(-1)^{(n-1)/2}\sin n\theta)+n(\cos(n-2)\theta-(-1)^{(n-1)/2}\sin(n-2)\theta)\\&+\frac{n(n-1)}{2!}(\cos(n-4)\theta+(-1)^{(n-1)/2}\sin(n-4)\theta)\\&+\frac{n(n-1)(n-2)}{3!}(\cos(n-6)\theta-(-1)^{(n-1)/2}\sin(n-6)\theta)+\ldots\\&\ldots+\frac{n!}{\left[\frac{(n-1)}{2}!\right]\left[\frac{(n+1)}{2}!\right]}(\cos\theta+\sin\theta)\qquad \text{n odd}\end{aligned}\quad\ldots 9.3$$

Note 10. Rational Values of $\cos^n\theta+\sin^n\theta$

10.1 n even, n/2 even

From eqn 9.1 reproduced below, the possible values of n are 4,8,12,16 …

$$2^{n-2}(\cos^n\theta+\sin^n\theta)=\cos n\theta+\frac{n(n-1)}{2!}\cos(n-4)\theta+\frac{n(n-1)(n-2)(n-3)}{4!}\cos(n-8)\theta+\ldots+1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}\ \ldots\ldots\ \text{n even, n/2 even} \quad \ldots\ldots 10.1$$

Table 10.1.1 shows the values of $n\theta$ which make $\cos n\theta$ rational. Examples are given for n=4, 8, 12 and 16. The angles in common for $0<\theta<90°$ are $15°,22.5°,30°,45°,60°,67.5°$ and $75°$.

For these angles the rhs of eqn 10.1 is rational for all n even, n/2 even

That is for $0<\theta<90°$, $\cos^n\theta+\sin^n\theta$ is rational for $\theta=15°,22.5°,30°,45°,60°,67.5°$ and $75°$ and n=4,8,12,16…..

Theta for which cos n theta are rational; n=1

	k=												
Theta	0	1	2	3	4	5	6	7	8	9	10	11	12
180 k +90	90	270	450	630	810	990	1170	1350	1530	1710	1890	2070	2250
360 k + 60	60	420	780	1140	1500	1860	2220	2580	2940	3300	3660	4020	4380
360k - 60	-60	300	660	1020	1380	1740	2100	2460	2820	3180	3540	3900	4260
180k +60	60	240	420	600	780	960	1140	1320	1500	1680	1860	2040	2220
180k-60	-60	120	300	480	660	840	1020	1200	1380	1560	1740	1920	2100
360k	0	360	720	1080	1440	1800	2160	2520	2880	3240	3600	3960	4320
180k	0	180	360	540	720	900	1080	1260	1440	1620	1800	1980	2160

Theta for which cos n theta are rational; n=4

Theta	k=0	1	2	3	4	5	6	7	8	9	10	11	12
180 k +90	22.5	67.5	112.5	157.5	202.5	247.5	292.5	337.5	382.5	427.5	472.5	517.5	562.5
360 k + 60	15	105	195	285	375	465	555	645	735	825	915	1005	1095
360k - 60	-15	75	165	255	345	435	525	615	705	795	885	975	1065
180k +60	15	60	105	150	195	240	285	330	375	420	465	510	555
180k-60	-15	30	75	120	165	210	255	300	345	390	435	480	525
360k	0	90	180	270	360	450	540	630	720	810	900	990	1080
180k	0	45	90	135	180	225	270	315	360	405	450	495	540

Theta for which cos n theta are rational; n=8

Theta	k=0	1	2	3	4	5	6	7	8	9	10	11	12
180 k +90	11.25	33.75	56.25	78.75	101.25	123.75	146.25	168.75	191.25	213.75	236.25	258.75	281.25
360 k + 60	7.5	52.5	97.5	142.5	187.5	232.5	277.5	322.5	367.5	412.5	457.5	502.5	547.5
360k - 60	-7.5	37.5	82.5	127.5	172.5	217.5	262.5	307.5	352.5	397.5	442.5	487.5	532.5
180k +60	7.5	30	52.5	75	97.5	120	142.5	165	187.5	210	232.5	255	277.5
180k-60	-7.5	15	37.5	60	82.5	105	127.5	150	172.5	195	217.5	240	262.5
360k	0	45	90	135	180	225	270	315	360	405	450	495	540
180k	0	22.5	45	67.5	90	112.5	135	157.5	180	202.5	225	247.5	270

common theta for n=8 and 4

0	15	22.5	30	45	60	67.5	75	90	105	112.5	120	135	150

Theta for which cos n theta are rational; n=12

Theta	k=0	1	2	3	4	5	6	7	8	9	10	11	12
180 k +90	7.5	22.5	37.5	52.5	67.5	82.5	97.5	112.5	127.5	142.5	157.5	172.5	187.5
360 k + 60	5	35	65	95	125	155	185	215	245	275	305	335	365
360k - 60	-5	25	55	85	115	145	175	205	235	265	295	325	355
180k +60	5	20	35	50	65	80	95	110	125	140	155	170	185
180k-60	-5	10	25	40	55	70	85	100	115	130	145	160	175
360k	0	30	60	90	120	150	180	210	240	270	300	330	360
180k	0	15	30	45	60	75	90	105	120	135	150	165	180

common theta for n=12, 8 and 4

0	15	22.5	30	45	60	67.5	75	90	105	112.5	120	135	150

Theta for which cos n theta are rational; n=16

Theta	k=0	1	2	3	4	5	6	7	8	9	10	11	12
180 k +90	5.625	16.875	28.125	39.375	50.625	61.875	73.125	84.375	95.625	106.875	118.125	129.375	140.625
360 k + 60	3.75	26.25	48.75	71.25	93.75	116.25	138.75	161.25	183.75	206.25	228.75	251.25	273.75
360k - 60	-3.75	18.75	41.25	63.75	86.25	108.75	131.25	153.75	176.25	198.75	221.25	243.75	266.25
180k +60	3.75	15	26.25	37.5	48.75	60	71.25	82.5	93.75	105	116.25	127.5	138.75
180k-60	-3.75	7.5	18.75	30	41.25	52.5	63.75	75	86.25	97.5	108.75	120	131.25
360k	0	22.5	45	67.5	90	112.5	135	157.5	180	202.5	225	247.5	270
180k	0	11.25	22.5	33.75	45	56.25	67.5	78.75	90	101.25	112.5	123.75	135

common theta for n=16, 12, 8 and 4

0	15	22.5	30	45	60	67.5	75	90	105	112.5	120	135	150

Table 10.1.1 Values of θ for which $\cos n\theta$ are rational. Examples for n=1, 4, 8, 12, 16

10.2 n even, n/2 odd
Eqn. 9.2 is reproduced below.

$$2^{n-2}(\cos^n\theta+\sin^n\theta)=n\cos(n-2)\theta+\frac{n(n-1)(n-2)}{3!}\cos(n-6)\theta$$
$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos(n-10)\theta+\ldots \quad \ldots 10.2$$
$$\ldots+1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}\ldots\ldots \quad \text{n even, n/2 odd}$$

Values of n >2 that satisfy equation 10.2 are 6,10,14,18 For the lhs of this equation to be rational $\cos 4\theta, \cos 8\theta, \cos 12\theta$ have to be rational. This is the same case as n even, n/2 even above in Note 10.1.

Therefore for $0<\theta<90°$ $\cos^n\theta+\sin^n\theta$ is rational for $\theta=15°,22.5°,30°,45°,60°,67.5°$ and $75°$ with n=2, 6,10,14,18…..

For n=2 the rhs is an integer n.

10.3 n odd

Eqn 9.3 is reproduced below.

$$2^{n-1}(\cos^n\theta+\sin^n\theta)=(\cos n\theta+(-1)^{(n-1)/2}\sin n\theta)+n(\cos(n-2)\theta-(-1)^{(n-1)/2}\sin(n-2)\theta)$$
$$+\frac{n(n-1)}{2!}(\cos(n-4)\theta+(-1)^{(n-1)/2}\sin(n-4)\theta)$$
$$+\frac{n(n-1)(n-2)}{3!}(\cos(n-6)\theta-(-1)^{(n-1)/2}\sin(n-6)\theta)+...$$
$$...+\frac{n!}{\left[\frac{(n-1)}{2}!\right]\left[\frac{(n+1)}{2}!\right]}(\cos\theta+\sin\theta) \qquad \text{n odd} \qquad ...10.3$$

Tables 10.3.1 (a) and (b) show that the common values of θ
for which $\cos n\theta$ and $\sin n\theta$ are simultaneously rational are 0,90,180,270, 360…..,k90 where k=0,1,2….

Therefore there is no rational value for $\cos^n\theta+\sin^n\theta$ for n odd for $0<\theta<90$.

Theta for which cos n theta is rational n = 1

	k=								
Theta	0	1	2	3	4	5	6	7	8
180 k +90	90.0	270.0	450.0	630.0	810.0	990.0	1170.0	1350.0	1530.0
360 k + 60	60.0	420.0	780.0	1140.0	1500.0	1860.0	2220.0	2580.0	2940.0
360k - 60	-60.0	300.0	660.0	1020.0	1380.0	1740.0	2100.0	2460.0	2820.0
180k +60	60.0	240.0	420.0	600.0	780.0	960.0	1140.0	1320.0	1500.0
180k-60	-60.0	120.0	300.0	480.0	660.0	840.0	1020.0	1200.0	1380.0
360k	0.0	360.0	720.0	1080.0	1440.0	1800.0	2160.0	2520.0	2880.0
180k	0.0	180.0	360.0	540.0	720.0	900.0	1080.0	1260.0	1440.0

Theta for which sin n theta is rational n= 1

	k=								
Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	180.0	360.0	540.0	720.0	900.0	1080.0	1260.0	1440.0
360 k + (90+60)	150.0	510.0	870.0	1230.0	1590.0	1950.0	2310.0	2670.0	3030.0
360k +30	30.0	390.0	750.0	1110.0	1470.0	1830.0	2190.0	2550.0	2910.0
180k +(90+60)	150.0	330.0	510.0	690.0	870.0	1050.0	1230.0	1410.0	1590.0
180k+(90-60)	30.0	210.0	390.0	570.0	750.0	930.0	1110.0	1290.0	1470.0
360k+90	90.0	450.0	810.0	1170.0	1530.0	1890.0	2250.0	2610.0	2970.0
360k+270	270.0	630.0	990.0	1350.0	1710.0	2070.0	2430.0	2790.0	3150.0

Common theta for rational sin n theta and cos n theta n=1 =90k

90	180	270	360	450	540	630	720	810	900

Theta for which cos n theta is rational n = 3

Theta	0	1	2	3	4	5	6	7	8
180 k +90	30.0	90.0	150.0	210.0	270.0	330.0	390.0	450.0	510.0
360 k + 60	20.0	140.0	260.0	380.0	500.0	620.0	740.0	860.0	980.0
360k - 60	-20.0	100.0	220.0	340.0	460.0	580.0	700.0	820.0	940.0
180k +60	20.0	80.0	140.0	200.0	260.0	320.0	380.0	440.0	500.0
180k-60	-20.0	40.0	100.0	160.0	220.0	280.0	340.0	400.0	460.0
360k	0.0	120.0	240.0	360.0	480.0	600.0	720.0	840.0	960.0
180k	0.0	60.0	120.0	180.0	240.0	300.0	360.0	420.0	480.0

Theta for which sin n theta is rational n = 3

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	60.0	120.0	180.0	240.0	300.0	360.0	420.0	480.0
360 k + (90+60)	50.0	170.0	290.0	410.0	530.0	650.0	770.0	890.0	1010.0
360k +30	10.0	130.0	250.0	370.0	490.0	610.0	730.0	850.0	970.0
180k +(90+60)	50.0	110.0	170.0	230.0	290.0	350.0	410.0	470.0	530.0
180k+(90-60)	10.0	70.0	130.0	190.0	250.0	310.0	370.0	430.0	490.0
360k+90	30.0	150.0	270.0	390.0	510.0	630.0	750.0	870.0	990.0
360k+270	90.0	210.0	330.0	450.0	570.0	690.0	810.0	930.0	1050.0

Common theta for rational sin n theta and cos n theta n=1 and 3 =90k

90	180	270	360	450	540	630	720	810	900

Table 10.3.1 (a) Rational values of $\cos n\theta$ and $\sin n\theta$ for n=1 and 3 and the common θ.

Theta for which cos n theta is rational n = 5

Theta	0	1	2	3	4	5	6	7	8
180 k +90	18.0	54.0	90.0	126.0	162.0	198.0	234.0	270.0	306.0
360 k + 60	12.0	84.0	156.0	228.0	300.0	372.0	444.0	516.0	588.0
360k - 60	-12.0	60.0	132.0	204.0	276.0	348.0	420.0	492.0	564.0
180k +60	12.0	48.0	84.0	120.0	156.0	192.0	228.0	264.0	300.0
180k-60	-12.0	24.0	60.0	96.0	132.0	168.0	204.0	240.0	276.0
360k	0.0	72.0	144.0	216.0	288.0	360.0	432.0	504.0	576.0
180k	0.0	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0

Theta for which sin n theta is rational n= 5

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	36.0	72.0	108.0	144.0	180.0	216.0	252.0	288.0
360 k + (90+60)	30.0	102.0	174.0	246.0	318.0	390.0	462.0	534.0	606.0
360k +30	6.0	78.0	150.0	222.0	294.0	366.0	438.0	510.0	582.0
180k +(90+60)	30.0	66.0	102.0	138.0	174.0	210.0	246.0	282.0	318.0
180k+(90-60)	6.0	42.0	78.0	114.0	150.0	186.0	222.0	258.0	294.0
360k+90	18.0	90.0	162.0	234.0	306.0	378.0	450.0	522.0	594.0
360k+270	54.0	126.0	198.0	270.0	342.0	414.0	486.0	558.0	630.0

Common theta for rational sin n theta and cos n theta n=1, 3 and 5 =90k

90	180	270	360	450	540	630	720	810	900

Theta for which cos n theta is rational n = 7

Theta	0	1	2	3	4	5	6	7	8
180 k +90	12.9	38.6	64.3	90.0	115.7	141.4	167.1	192.9	218.6
360 k + 60	8.6	60.0	111.4	162.9	214.3	265.7	317.1	368.6	420.0
360k - 60	-8.6	42.9	94.3	145.7	197.1	248.6	300.0	351.4	402.9
180k +60	8.6	34.3	60.0	85.7	111.4	137.1	162.9	188.6	214.3
180k-60	-8.6	17.1	42.9	68.6	94.3	120.0	145.7	171.4	197.1
360k	0.0	51.4	102.9	154.3	205.7	257.1	308.6	360.0	411.4
180k	0.0	25.7	51.4	77.1	102.9	128.6	154.3	180.0	205.7

Theta for which sin n theta is rational n= 7

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	25.7	51.4	77.1	102.9	128.6	154.3	180.0	205.7
360 k + (90+60)	21.4	72.9	124.3	175.7	227.1	278.6	330.0	381.4	432.9
360k +30	4.3	55.7	107.1	158.6	210.0	261.4	312.9	364.3	415.7
180k +(90+60)	21.4	47.1	72.9	98.6	124.3	150.0	175.7	201.4	227.1
180k+(90-60)	4.3	30.0	55.7	81.4	107.1	132.9	158.6	184.3	210.0
360k+90	12.9	64.3	115.7	167.1	218.6	270.0	321.4	372.9	424.3
360k+270	38.6	90.0	141.4	192.9	244.3	295.7	347.1	398.6	450.0

Common theta for rational sin n theta and cos n theta n=1, 3, 5and 7 =90k

90	180	270	360	450	540	630	720	810	900

Table 10.3.1 (b) Rational values of $\cos n\theta$ and $\sin n\theta$ for n=5 and 7 and the common θ.

As a summary this Note shows that for $0° < \theta < 90°$ $\cos^n \theta + \sin^n \theta$ is only rational for $\theta = 15°, 22.5°, 30°, 45°, 60°, 67.5°$ and $75°$ and for n even.

For all other θ where $0° < \theta < 90°$ and for all other n, $\cos^n \theta + \sin^n \theta$ is irrational.

Note 11. Rational values of $(\cos^n\theta+\sin^n\theta)^2$

Note 10 has shown that for $0° < \theta < 90°$, $(\cos^n\theta+\sin^n\theta)$ is rational for n even and $\theta = 15, 22.5, 30, 45, 60, 67.5$ and $75°$. For n odd there are no rational values over the same angular range.
In this section we find the rational values of $(\cos^n\theta+\sin^n\theta)^2$ over the same angular range.

By expansion,

$$(\cos^n\theta+\sin^n\theta)^2 = \cos^{2n}\theta+\sin^{2n}\theta+2\cos^n\theta\sin^n\theta$$

Since

$$2\cos^n\theta\sin^n\theta = 2(\cos\theta\sin\theta)^n = 2(\sin 2\theta/2)^n = \frac{\sin^n 2\theta}{2^{n-1}}$$

$$= \cos^{2n}\theta+\sin^{2n}\theta+\sin^n 2\theta/2^{n-1}$$

11.1 n even

The first part of the expansion is $\cos^{2n}\theta+\sin^{2n}\theta \equiv \cos^k\theta+\sin^k\theta$, k even

and as above is rational for $\theta = 15, 22.5, 30, 45, 60, 67.5$ and $75°$.

$\sin^n 2\theta$ is given by the expansion of eqn. 8.1 reproduced below,

$$(-1)^{n/2}2^{n-1}\sin^n 2\theta = \cos n2\theta - n\cos(n-2)2\theta + \frac{n(n-1)}{2!}\cos(n-4)2\theta - \ldots$$
$$\ldots + (-1)^{n/2}1/2\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)} \ldots\ldots \text{ n even}$$

This is rational for rational $\cos 2n\theta, \cos(2n-4)\theta, \cos(2n-8)\theta, \cos(2n-12)\theta\ldots$ for n even i.e n=4,8,12,16….

These are the same values of n as given in Table 10.1.1 for $(\cos^n\theta+\sin^n\theta)$.

$\sin^n 2\theta$ is therefore rational for $\theta = 15, 22.5, 30, 45, 60, 67.5$ and $75°$.

The values of θ which are common for rational $\cos^{2n}\theta+\sin^{2n}\theta$ and $\sin^n 2\theta$ i.e. $(\cos^n\theta+\sin^n\theta)^2$ are therefore $\theta = 15, 22.5, 30, 45, 60, 67.5$ and $75°$ i.e. $(\cos^n\theta+\sin^n\theta)^2$ for n even is rational for $\theta = 15, 22.5, 30, 45, 60, 67.5$ and $75°$

11.2 n odd

For the first part of the expansion $\cos^{2n}\theta + \sin^{2n}\theta \equiv \cos^k\theta + \sin^k\theta$ k is even as before for n odd, therefore rational values of this are as before $\theta = 15, 22.5, 30, 45, 60, 67.5 \text{ and } 75°$.

$\sin^n 2\theta$ for n odd is given by eqn 8.2 reproduced below for $\theta \Rightarrow 2\theta$

$$(-1)^{(n-1)/2} 2^{n-1} \sin^n 2\theta = \sin n2\theta - n\sin(n-2)2\theta + \frac{n(n-1)}{2!}\sin(n-4)2\theta - ...$$
$$... + (-1)^{(n-1)/2} \frac{n!}{(\frac{n-1}{2}!)(\frac{n+1}{2}!)} \sin 2\theta..... \text{ n odd}$$

This is rational for rational $\sin 2n\theta, \sin(2n-4)\theta, \sin(2n-8)\theta, \sin(2n-12)\theta....$ for n odd i.e for n=2,6,10,14…

Table 10.1.2 shows that this corresponds to $\theta = 15, 45 \text{ and } 75°$.

Therefore the θ in common for $\cos^{2n}\theta + \sin^{2n}\theta$ and $\sin^n 2\theta$ for n odd are $\theta = 15, 45 \text{ and } 75°$ i.e. $(\cos^n\theta + \sin^n\theta)^2$ for n odd is rational for $\theta = 15, 45 \text{ and } 75°$

Theta for which sin n theta is rational n= 12n=2

	k=								
Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	90.0	180.0	270.0	360.0	450.0	540.0	630.0	720.0
360 k + (90+60)	75.0	255.0	435.0	615.0	795.0	975.0	1155.0	1335.0	1515.0
360k +30	15.0	195.0	375.0	555.0	735.0	915.0	1095.0	1275.0	1455.0
180k +(90+60)	75.0	165.0	255.0	345.0	435.0	525.0	615.0	705.0	795.0
180k+(90-60)	15.0	105.0	195.0	285.0	375.0	465.0	555.0	645.0	735.0
360k+90	45.0	225.0	405.0	585.0	765.0	945.0	1125.0	1305.0	1485.0
360k+270	135.0	315.0	495.0	675.0	855.0	1035.0	1215.0	1395.0	1575.0

Theta for which sin n theta is rational n = :2n=6

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	30.0	60.0	90.0	120.0	150.0	180.0	210.0	240.0
360 k + (90+60)	25.0	85.0	145.0	205.0	265.0	325.0	385.0	445.0	505.0
360k +30	5.0	65.0	125.0	185.0	245.0	305.0	365.0	425.0	485.0
180k +(90+60)	25.0	55.0	85.0	115.0	145.0	175.0	205.0	235.0	265.0
180k+(90-60)	5.0	35.0	65.0	95.0	125.0	155.0	185.0	215.0	245.0
360k+90	15.0	75.0	135.0	195.0	255.0	315.0	375.0	435.0	495.0
360k+270	45.0	105.0	165.0	225.0	285.0	345.0	405.0	465.0	525.0

Common theta for rational sin n theta n=1 and 3

15	45	75	90	105	135	165	180		

Theta for which sin n theta is rational n= 52n=10

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	18.0	36.0	54.0	72.0	90.0	108.0	126.0	144.0
360 k + (90+60)	15.0	51.0	87.0	123.0	159.0	195.0	231.0	267.0	303.0
360k +30	3.0	39.0	75.0	111.0	147.0	183.0	219.0	255.0	291.0
180k +(90+60)	15.0	33.0	51.0	69.0	87.0	105.0	123.0	141.0	159.0
180k+(90-60)	3.0	21.0	39.0	57.0	75.0	93.0	111.0	129.0	147.0
360k+90	9.0	45.0	81.0	117.0	153.0	189.0	225.0	261.0	297.0
360k+270	27.0	63.0	99.0	135.0	171.0	207.0	243.0	279.0	315.0

Common theta for rational sin n theta and cos n theta n=1, 3 and 5

15	45	75	90	105	135	165	180		

Theta for which sin n theta is rational n= 72n=14

Theta	0	1	2	3	4	5	6	7	8
180 k	0.0	12.9	25.7	38.6	51.4	64.3	77.1	90.0	102.9
360 k + (90+60)	10.7	36.4	62.1	87.9	113.6	139.3	165.0	190.7	216.4
360k +30	2.1	27.9	53.6	79.3	105.0	130.7	156.4	182.1	207.9
180k +(90+60)	10.7	23.6	36.4	49.3	62.1	75.0	87.9	100.7	113.6
180k+(90-60)	2.1	15.0	27.9	40.7	53.6	66.4	79.3	92.1	105.0
360k+90	6.4	32.1	57.9	83.6	109.3	135.0	160.7	186.4	212.1
360k+270	19.3	45.0	70.7	96.4	122.1	147.9	173.6	199.3	225.0

Common theta for rational sin n theta and cos n theta n=1, 3, 5and 7

15	45	75	90	105	135	165	180		

Table 11.1.2 θ **for which** $\sin^n 2\theta$ for n odd is rational.

Note 12

Proof that if

$(\cos^n\theta+\sin^n\theta)^2$ is irrational then $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational.

(n = +ve integer)

Let $\alpha=(\cos^n\theta+\sin^n\theta)^2$ which is irrational.

α can be written,

$\alpha=\alpha^{1/n}\alpha^{1/n}\alpha^{1/n}\alpha^{1/n}$..............$n$ times $=(\alpha^{1/n})^n$.......4.1

By way of contradiction let $\alpha^{1/n}$ be rational.

Then the rhs of equation 4.1 is rational, since if

$\alpha^{1/n}=a/b$

with a, b integers,

then

$(\alpha^{1/n})^n=a^n/b^n=c/d$

is rational where c, d are integers.

But this is a contradiction since α is irrational.

Therefore the theorem is proved.

Note 11 has shown that except for $\theta=15,22.5,30,45,60,67.5$ and $75°$ for n even and $\theta=15$ and $45°$ for n odd, $(\cos^n\theta+\sin^n\theta)^2$ is irrational. Therefore the above note shows that $(\cos^n\theta+\sin^n\theta)^{2/n}$ is also irrational.

Note 13. For rational $(\cos^n\theta+\sin^n\theta)^2$ showing that $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational.

Since $(\cos^n\theta+\sin^n\theta)^2$ is symmetric about $45°$ i.e. $(\cos^n 15+\sin^n 15)^2$ $\equiv(\cos^n 75+\sin^n 75)^2$ we only have to show $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational for $\theta=15{,}22.5{,}30$ and $45°$ for n even and $\theta=15$ and $45°$ for n odd. i.e we have to show that for all n $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational for $\theta=15$ and $45°$ whereas for n even only it has to be shown that $(\cos^n\theta+\sin^n\theta)^{2/n}$ is irrational for $\theta=22.5$ and $30°$. This is shown below.

13.1 $\theta=15°$

Substituting $\theta=15°$ for n=2 into eqn 7.1 and 8.1 gives,

$$\cos^2 15=\frac{(2+\sqrt{3})}{4}\Rightarrow\cos 15=\left[\frac{(2+\sqrt{3})}{4}\right]^{1/2}$$

$$\sin^2 15=\frac{(2-\sqrt{3})}{4}\Rightarrow\sin 15=\left[\frac{(2-\sqrt{3})}{4}\right]^{1/2}$$

Now,

$$(\cos^n 15+\sin^n 15)^2 = \cos^{2n}15+\sin^{2n}15+\sin^n 30/2^{n-1}$$

where,

$$\sin^n 30=\frac{1}{2^n},$$

$$\therefore\sin^n 30/2^{n-1}=\frac{1}{2^n}.\frac{1}{2^{n-1}}=\frac{2}{2^{2n}}=\frac{2}{4^n}.$$

and,

$$\cos^{2n}15+\sin^{2n}15=\frac{(2+\sqrt{3})^n+(2-\sqrt{3})^n}{4^n}=\frac{B_n}{4^n}=S_n\,.$$

$$\therefore(\cos^n 15+\sin^n 15)^2=\frac{(B_n+2)}{4^n}=S_n+2/4^n=R_n$$

Using the binomial expansion for n integer,

$$(2\pm\sqrt{3})^n=(2)^n\pm\left[\tfrac{n}{1}\right](2)^{n-1}\left(\sqrt{3}\right)+\left[\tfrac{n}{2}\right](2)^{n-2}\left(\sqrt{3}\right)^2\pm\left[\tfrac{n}{3}\right](2)^{n-3}\left(\sqrt{3}\right)^3+\ldots+(\pm 1)^n\left(\sqrt{3}\right)^n$$

$$(2+\sqrt{3})^n+(2-\sqrt{3})^n$$

$$=B_n=2(2)^n+2\left[\tfrac{n}{2}\right]2^{n-2}(\sqrt{3})^2+2\left[\tfrac{n}{4}\right]2^{n-4}(\sqrt{3})^4+2\left[\tfrac{n}{6}\right]2^{n-6}(\sqrt{3})^6+\ldots+2(\sqrt{3})^n\quad\text{n even}$$

$$+0\qquad\text{n odd}$$

where $\lfloor{}^{n}_{j}\rfloor$ is the binomial coefficient $\frac{n!}{j!(n-j)!}$ which is an integer.

It can be seen from this expansion that for all n, B_n is an integer since the powers of $\sqrt{3}$ are always even. Therefore S_n is always rational as Note 10 showed.

Some values of B_n are shown in below.

n	B_n	F_n
1	4	6
2	14	4
3	52	3.77976
4	194	3.74166
5	724	3.73411
6	2702	3.73251

Table 13.1.1 Values of B_n and F_n .

Showing the nth root of R_n is irrational.

$$(R_n)^{1/n} = (\cos^n 15 + \sin^n 15)^{2/n} = \frac{(B_n + 2)^{1/n}}{4} = \frac{F_n}{4}$$

Since B_n and $(\mathrm{B_n} + 2)$ are integers, corollary 2 of Note 5 states that $(B_n + 2)^{1/n}$ is either integer or irrational.

But Table 13.1.1 shows that F_n is not an integer since, $3 < F_n < 4 \;\; \forall n > 2$. F_n is therefore irrational.

For large n ,
$F_n \Rightarrow (2 + \sqrt{3}) = 3.73205$.

Therefore $F_n / 4 = (\cos^n 15 + \sin^n 15)^{2/n}$ is irrational for n>2.

For n=2 we get the familiar identity,

$$F_2 / 4 = (\cos^2 15 + \sin^2 15)^{2/2} = 1$$

13.2 $\theta = 22.5°$, n even.

Substituting $\theta = 22.5°$ for n=2 into eqn 7.1 and 8.1 gives,

$$\cos^2 22.5 = \frac{(\sqrt{2}+1)}{2\sqrt{2}} \Rightarrow \cos 22.5 = \left[\frac{(\sqrt{2}+1)}{2\sqrt{2}}\right]^{1/2}$$

$$\sin^2 22.5 = \frac{(\sqrt{2}-1)}{2\sqrt{2}} \Rightarrow \sin 22.5 = \left[\frac{(\sqrt{2}-1)}{2\sqrt{2}}\right]^{1/2}$$

Now,

$$(\cos^n 22.5 + \sin^n 22.5)^2 = \cos^{2n} 22.5 + \sin^{2n} 22.5 + \sin^n 45 / 2^{n-1}$$

where,

$$\sin^n 45 = \frac{1}{(\sqrt{2})^n},$$

$$\therefore \sin^n 45/2^{n-1} = \frac{1}{(\sqrt{2})^n}.\frac{1}{2^{n-1}} = \frac{2}{2^{3n/2}}.$$

and

$$\cos^{2n} 22.5 + \sin^{2n} 22.5 = \frac{(\sqrt{2}+1)^n + (\sqrt{2}-1)^n}{2^{3n/2}} = \frac{B_n}{2^{3n/2}} = S_n,$$

$$\therefore (\cos^n 22.5 + \sin^n 22.5)^2 = \frac{(B_n+2)}{2^{3n/2}} = S_n + 2/2^{3n/2} = R_n.$$

Again using the binomial expansion

$$(\sqrt{2}\pm 1)^n = (\sqrt{2})^n \pm \left[\tfrac{n}{1}\right](\sqrt{2})^{n-1}(1) + \left[\tfrac{n}{2}\right](\sqrt{2})^{n-2}(1)^2 \pm \left[\tfrac{n}{3}\right](\sqrt{2})^{n-3}(1)^3 + \ldots + (\pm 1)^n (1)^n$$

$$(\sqrt{2}+1)^n + (\sqrt{2}-1)^n$$
$$= B_n = 2(\sqrt{2})^n + 2\left[\tfrac{n}{2}\right](\sqrt{2})^{n-2} + 2\left[\tfrac{n}{4}\right](\sqrt{2})^{n-4} + 2\left[\tfrac{n}{6}\right](\sqrt{2})^{n-6} + \ldots + 2 \quad \text{n even} \quad \textbf{…13.2.1}$$

where $\left[\tfrac{n}{j}\right]$ is the binomial coefficient $\frac{n!}{j!(n-j)!}$ which is an integer.

It can be seen from this expansion that for n even B_n is an integer since the powers of $\sqrt{2}$ are always even.

Some values of B_n are shown in below.

n	B_n	F_n
2	6	2.8284= $2^{3/2}$
4	34	2.4495
6	198	2.4183
8	1154	2.4147
10	6726	2.4143
20	45239074	2.4142

Table 13.2.1 Values of B_n and F_n.

Showing the nth root of R_n is irrational.

$$(R_n)^{1/n} = (\cos^n 22.5 + \sin^n 22.5)^{2/n} = \frac{(B_n+2)^{1/n}}{2^{3/2}} = \frac{F_n}{2^{3/2}}$$

Since B_n and $(\mathrm{B_n}+2)$ are integers, corollary 2 of Note 5 states that $F_n = (B_n+2)^{1/n}$ is either an integer or irrational.

But Table 13.2.1 shows that F_n is not an integer since, $2 < F_n < 3 \ \ \forall n > 2$. F_n is therefore irrational.

For large n,
$F_n \Rightarrow (\sqrt{2}+1) = 2.4142$.

Let,
$$(R_n)^{1/n} = \frac{(B_n+2)^{1/n}}{2^{3/2}} = \frac{(A_n)^{1/n}}{2\sqrt{2}}$$

If $(A_n)^{1/n}$ is irrational then $(R_n)^{1/n}$ is irrational unless $(A_n)^{1/n}$ has $2\sqrt{2}$ as a 'factor' (e.g $2\pi/3$ is irrational so is π but $\frac{2\pi/3}{\pi} = 2/3$ which is rational.)

Case 1(a) Integer Factor

By way of contradiction assume $(A_n)^{1/n} = Z_n 2\sqrt{2}$ where Z_n is an integer factor.

Then $A_n = (Z_n)^n (2\sqrt{2})^n$

Since Z_n is an integer so is $(Z_n)^n = E_n$

$\therefore E_n = \frac{A_n}{(2\sqrt{2})^n}$ is an integer, n even.

But Table 13.2.2 shows that $E_n < 1 \ \ \forall n \text{ even} > 2$. Therefore there is a contradiction and the assumption is false.

n even	$A_n = B_n + 2$	$(2\sqrt{2})^n$	E_n
2	6	8	1.0000
4	34	64	0.5625
6	198	512	0.3906
8	1154	4096	0.2822
10	6726	32768	0.2053
30	45239074	1073741824	0.0086

Table 13.2.2 Values of E_n.

For large n,
$E_n \Rightarrow (1/(2\sqrt{2}) + 1/2)^n = 0.854^{\mathrm{n}}$ which $\Rightarrow 0$ as $n \Rightarrow \infty$.

Case 1(b) Rational Factor.

By way of contradiction assume $(A_n)^{1/n} = Q_n(2\sqrt{2})$ where Q_n is a rational factor.
Then $A_n = (Q_n)^n (2\sqrt{2})^n$

$(Q_n)^n$ equals $\left[\frac{L_n}{M_n}\right]^n$ where L_n and M_n are integers.

$\therefore$ for A_n to be integer,

$$M_n = \frac{(2\sqrt{2})}{T_n},$$

$$\therefore A_n = (L_n T_n)^n = (P_n)^n \text{ where } P_n \text{ is an integer.}$$

$$\therefore P_n = (A_n)^{1/n}$$

But Table 13.2.3 shows that P_n is not an integer. Therefore there is a contradiction and the assumption is incorrect.

n even	A_n	P_n
2	6	2.8284= $2^{3/2}$
4	34	2.4495
6	198	2.4183
8	1154	2.4147
10	6726	2.4143
30	45239074	2.4142

Table 13.2.3 Values of P_n.

For $n \Rightarrow \infty, \quad P_n \Rightarrow (\sqrt{2}+1) = 2.4142$

Therefore since $F_n = (A_n)^{1/n}$ is irrational and doesn't have $2^{3/2} = 2\sqrt{2}$ as an integral or rational factor, $F_n / 2^{3/2} = (R_n)^{1/n} = (\cos^n 22.5 + \sin^n 22.5)^{2/n}$ is irrational for n even >2.

For n=2 we get the familiar identity,

$$F_2 / 2^{3/2} = (\cos^2 22.5 + \sin^2 22.5)^{2/2} = 1$$

13.3 $\theta = 30°$ n even.

$$\cos^n 30 + \sin^n 30 = (\sqrt{3}/2)^n + (1/2)^n = \frac{((\sqrt{3})^n + 1)}{2^n}$$

$$\therefore\ (\cos^n 30 + \sin^n 30)^2 = \frac{((\sqrt{3})^{2n} + 2(\sqrt{3})^n + 1)}{4^n} = \frac{B_n}{4^n} = R_n$$

As n is even, B_n is an integer since the powers of $\sqrt{3}$ are always even.
Some values of B_n are shown in below.

n	B_n	F_n
2	16	4
4	100	3.16228
6	784	3.03659
8	6724	3.00922
10	59536	3.00247
20	3486902500	3.00001

Table 13.3.1 Values of B_n and F_n .

Showing the nth root of R_n is irrational.

$$(R_n)^{1/n} = (\cos^n 30 + \sin^n 30)^{2/n} = \frac{(B_n)^{1/n}}{4} = \frac{F_n}{4}$$

Since B_n is an integer, corollary 2 of Note 5 states that $F_n = (B_n)^{1/n}$ is either an integer or irrational.

But Table 13.3.1 shows that F_n is not an integer since, $3 < F_n < 4\ \forall n > 2$. F_n is therefore irrational.

For large n ,
$F_n \Rightarrow ((\sqrt{3})^{2n})^{1/n} = 3$.

Therefore $F_n / 4 = (\cos^n 30 + \sin^n 30)^{2/n}$ is irrational for n even >2.

For n=2 we get the familiar identity,
$F_2 / 4 = (\cos^2 30 + \sin^2 30)^{2/2} = 1$

13.4 $\theta = 45°$ for all n.

$\cos^n 45 + \sin^n 45 = (1/\sqrt{2})^n + (1/\sqrt{2})^n = 2(1/\sqrt{2})^n = 2/(2)^{n/2}$

$\therefore (\cos^n 45 + \sin^n 45)^2 = \frac{4}{2^n} = R_n$

which is rational.

$(R_n)^{1/n} = \frac{4^{1/n}}{2}$

n	$4^{1/n}$
1	4
2	2
3	1.58740
4	1.41421
5	1.31951
20	1.07177

Table 13.4.1 Values of B_n

Corollary 2 of Note 5 states that $4^{1/n}$ is either an integer or irrational.

But Table 13.4.1 shows that for n>2 $4^{1/n}$ is not an integer and is therefore irrational.

For large n,
$4^{1/n} \Rightarrow 1$.

Therefore $(R_n)^{1/n} = 4^{1/n}/2 = (\cos^n 45 + \sin^n 45)^{2/n}$ is irrational for n >2.

For n=2 we get the familiar identity,

$4^{1/2}/2 = (\cos^2 45 + \sin^2 45)^{2/2} = 1$

Appendix 1 History of Trigonometric Functions.
(e.g.http://wwwgroups.dcs.stand.ac.uk/~history/HistTopics/Trigonometric_functions.html).

The following is lifted from the above web reference and shows that Fermat had access to sine and cosine tabulations about the time of the formulation of his Last Theorem.

The Arabs worked with sines and cosines and by 980 Abu'l-Wafa knew that

$$\sin 2x = 2 \sin x \cos x$$

although it could have easily have been deduced from Ptolemy's formula $\sin(x + y) = \sin x \cos y + \cos x \sin y$ with $x = y$.

The Hindu word *jya* for the sine was adopted by the Arabs who called the sine *jiba*, a meaningless word with the same sound as *jya*. Now *jiba* became *jaib* in later Arab writings and this word does have a meaning, namely a 'fold'. When European authors translated the Arabic mathematical works into Latin they translated jaib into the word sinus meaning fold in Latin. In particular Fibonacci's use of the term *sinus rectus arcus* soon encouraged the universal use of sine.

Chapters of Copernicus's book giving all the trigonometry relevant to astronomy was published in 1542 by Rheticus. Rheticus also produced substantial tables of sines and cosines which were published after his death. In 1533 Regiomontanus's work *De triangulis omnimodis* was published. This contained work on planar and spherical trigonometry originally done much earlier in about 1464. The book is particularly strong on the sine and its inverse.

The term sine certainly was not accepted straight away as the standard notation by all authors. In times when mathematical notation was in itself a new idea many used their own notation. Edmund Gunter was the first to use the abbreviation sin in 1624 in a drawing. The first use of sin in a book was in 1634 by the French mathematician Hérigone while Cavalieri used *Si* and Oughtred *S*.

The cosine follows a similar course of development in notation as the sine. Viète used the term sinus residuae for the cosine, Gunter (1620) suggested co-sinus. The notation Si.2 was used by Cavalieri, s co arc by Oughtred and S by Wallis.

References

1. BBC TV Horizon script on FLT. See www.bbc.co.uk/horizon
2 I Niven. Numbers Rational and Irrational. 1963.
3. E.W.Hobson. A Treatise on Plane and Advanced Trigonometry.New York: Dover Publication 1957.