A Geometric Visualisation of Fermat's Last Theorem

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Abstract

Fermat's Last Theorem asserts that the equation

$$x^n + y^n = z^n$$

has no non-zero integer solutions when $n > 2$. Its celebrated proof, due to Wiles [1], lies deep within modern number theory, but the statement itself invites a simple geometric question. Pythagorean triples give rational points on the unit circle $x^2 + y^2 = 1$. What becomes of these rational directions when the exponent is increased?

We visualise this question using the family of normalised Fermat curves

$$x^n + y^n = 1,$$

and follow rays through rational Pythagorean points as they intersect the higher-order curves. The resulting construction gives an elementary geometric interpretation of the change from $n = 2$ to $n > 2$. We then show how the picture can be made rigorous for an infinite structured family of Pythagorean triples, for which every corresponding projected intersection on a higher Fermat curve is irrational. The result provides a visual complement to the classical theory rather than an alternative proof of Fermat's Last Theorem.

The Fermat Visualisation

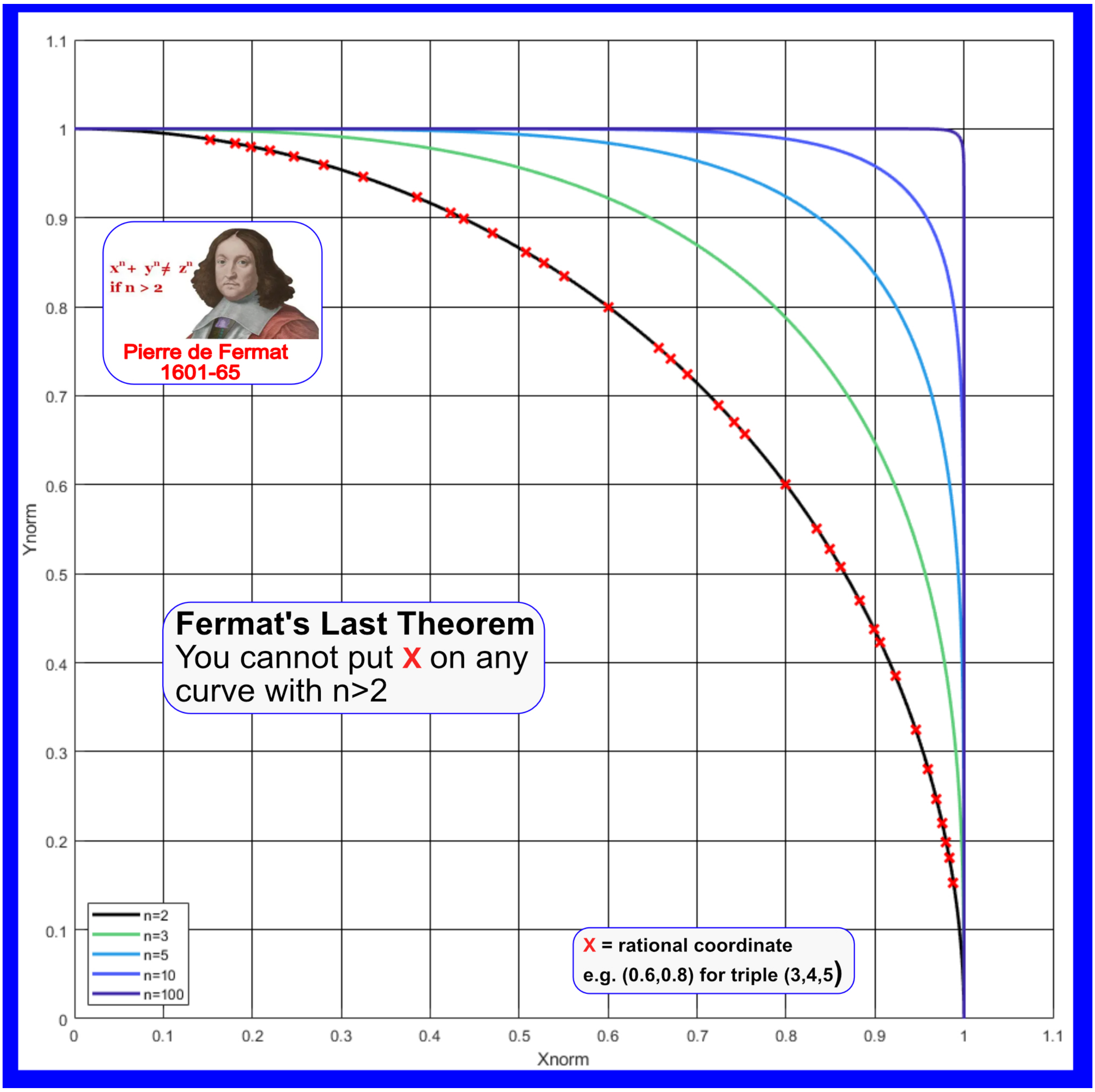

Figure 1: **The Fermat graphic.** The curves show $x^n + y^n = 1$ for several integer exponents $n \geq 2$. The red crosses are selected rational points on the unit circle $x^2 + y^2 = 1$, obtained by normalising Pythagorean triples. For $n > 2$, Fermat's Last Theorem is equivalently the statement that no non-trivial rational point lies on $x^n + y^n = 1$. The graphic therefore turns the theorem into a geometric question: what happens to rational Pythagorean directions when they are projected from the circle to the higher-order curves?

Pythagorean triples provide an immediate bridge between arithmetic and geometry. If

$$X^2 + Y^2 = Z^2,$$

then division by Z^2 places the rational point

$$\left(\frac{X}{Z}, \frac{Y}{Z}\right)$$

on the unit circle. Thus the familiar triples $(3, 4, 5)$, $(5, 12, 13)$, and infinitely many others appear as rational points on the $n = 2$ curve in Figure 1.

Now increase the exponent. The circle deforms into the family of curves

$$x^n + y^n = 1, \qquad n > 2.$$

The red crosses in Figure 1 remain fixed, while the curves move outward towards the boundary of the unit square. Fermat's Last Theorem says, in this normalised picture, that no non-trivial rational cross can be placed on any of these higher curves.

This suggests a particularly direct way of looking at the theorem. Rather than beginning with integer powers, begin with a rational point on the Pythagorean circle, draw the ray from the origin through that point, and ask where the same ray meets a higher Fermat curve. The geometry is elementary. The arithmetic of the new intersection is where Fermat's problem reappears.

From Pythagoras to Fermat

The case $n = 2$ is exceptional in a very visible way. The equation

$$X^2 + Y^2 = Z^2$$

has infinitely many integer solutions. Dividing by Z^2 gives

$$\left(\frac{X}{Z}\right)^2 + \left(\frac{Y}{Z}\right)^2 = 1,$$

so every Pythagorean triple determines a rational point on the unit circle. Conversely, a non-trivial rational point on the unit circle can be cleared of denominators to produce an integer Pythagorean triple. Thus Pythagorean triples and non-trivial rational points on

$$x^2 + y^2 = 1$$

are two ways of viewing the same arithmetic structure.

For example, the primitive triple

$$(3, 4, 5)$$

gives the two symmetric rational points

$$\left(\frac{3}{5}, \frac{4}{5}\right) \qquad \text{and} \qquad \left(\frac{4}{5}, \frac{3}{5}\right).$$

More generally, Euclid's classical parametrisation [2]

$$X = r^2 - s^2, \qquad Y = 2rs, \qquad Z = r^2 + s^2$$

produces infinitely many such points. The circle in Figure 1 is therefore not merely a geometric curve: it carries an infinite arithmetic population of rational points.

Now consider Fermat's equation

$$X^n + Y^n = Z^n, \qquad n > 2.$$

If a non-zero integer solution existed, division by Z^n would give

$$\left(\frac{X}{Z}\right)^n + \left(\frac{Y}{Z}\right)^n = 1.$$

Hence a Fermat solution would appear in the normalised picture as a non-trivial rational point on

$$x^n + y^n = 1.$$

The converse is equally immediate: clearing the denominators of a rational point on this curve gives an integer solution of Fermat's equation. Fermat's Last Theorem may therefore be expressed geometrically as the assertion that, for every integer $n > 2$, the curve

$$x^n + y^n = 1$$

contains no non-trivial rational point.

This normalisation puts the Pythagorean and Fermat problems into the same picture. Instead of comparing equations of different sizes, we compare a family of curves contained in the unit square. In the first quadrant they are the unit curves of the L_n-norm,

$$\|(x, y)\|_n = (x^n + y^n)^{1/n}.$$

For $n = 2$ the curve is the familiar quarter-circle. As n increases it moves outward, approaching the boundary of the unit square. Yet the rational Pythagorean points remain fixed on the circle.

The natural question is therefore not simply where the higher curves lie, but what happens if we follow one of the rational directions already supplied by a Pythagorean triple. A ray from the origin through a rational point (a, b) on the unit circle must meet every higher Fermat curve at exactly one point. The location of that new intersection can be found explicitly, and its arithmetic character will provide the link between the geometry of Figure 1 and the special families considered later.

The Geometry Behind the Graphic

Figure 1 contains more information than first appears. Every red cross determines a unique ray from the origin. Since each normalised Fermat curve

$$x^n + y^n = 1, \qquad n \geq 2,$$

is convex in the first quadrant, every such ray intersects each higher Fermat curve exactly once. Thus a single rational Pythagorean direction generates a sequence of intersection points, one on each member of the Fermat family.

This observation transforms Fermat's Last Theorem into a geometric question. The direction of the ray never changes; only the point at which it meets the curve changes. The problem therefore becomes one of understanding the arithmetic of these new intersection points.

Suppose that

$$(a, b),$$

with

$$a^2 + b^2 = 1,$$

is the rational point obtained by normalising a primitive Pythagorean triple. Let

$$(a', b')$$

denote the point where the same ray meets the higher Fermat curve

$$x^n + y^n = 1, \qquad n > 2.$$

Since both points lie on the same ray, the corresponding triangles are similar. Consequently the ratio of the coordinates is unchanged,

$$\frac{b'}{a'} = \frac{b}{a},$$

so the higher-order point is simply a scaled version of the original Pythagorean point.

The geometry therefore reduces to determining a single scale factor. Once this factor is known, every coordinate on every Fermat curve is determined immediately from the corresponding rational point on the unit circle.

Figure 2 illustrates this construction for a single Pythagorean triple. Beginning with the rational point on the unit circle, the ray is extended until it intersects the higher-order Fermat curve. The intersection generally lies further from the origin, reflecting the outward movement of the curves visible in Figure 1.

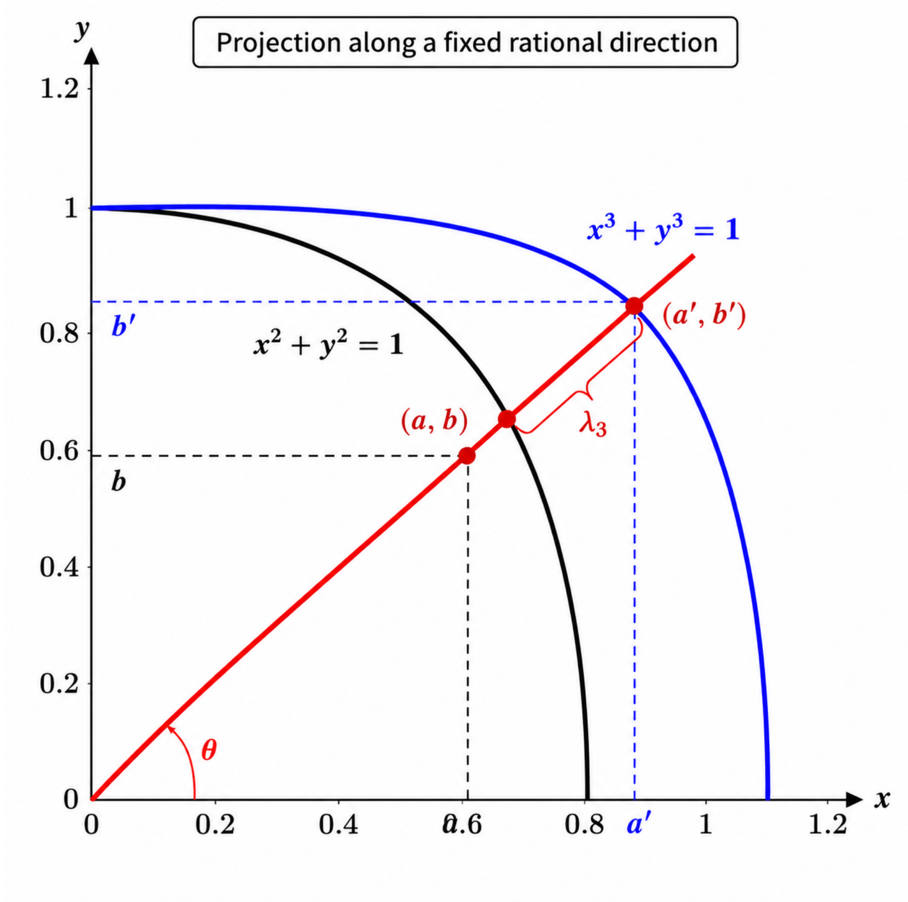

Figure 2: **Projection of a rational Pythagorean point onto a higher Fermat curve.** A ray through the rational point (a, b) on the unit circle intersects every curve $x^n + y^n = 1$ at a unique point (a', b'). The two points are related by a single scale factor because they lie on the same ray.

The existence of this common scaling immediately suggests that the coordinates (a', b') should possess a simple closed form. Deriving this expression is the key step in the geometric construction.

The Projection Principle

The construction described in the previous section determines the coordinates of the intersection point explicitly.

Let

$$(a, b),$$

be a rational point on the unit circle,

$$a^2 + b^2 = 1,$$

and let

$$(a', b')$$

be the corresponding point on the higher Fermat curve

$$x^n + y^n = 1, \qquad n > 2.$$

Since both points lie on the same ray from the origin, the triangles are similar, giving

$$\frac{b'}{a'} = \frac{b}{a},$$

or equivalently,

$$b' = \frac{b}{a}a'.$$

Substituting this relation into the equation of the Fermat curve gives

$$(a')^n + \left(\frac{b}{a}a'\right)^n = 1,$$

so that

$$(a')^n \left(1 + \frac{b^n}{a^n}\right) = 1.$$

Hence

$$(a')^n = \frac{a^n}{a^n + b^n},$$

and therefore

$$\boxed{a' = \frac{a}{(a^n + b^n)^{1/n}}}$$

Similarly,

$$\boxed{b' = \frac{b}{(a^n + b^n)^{1/n}}}$$

These two expressions completely determine the intersection of every rational Pythagorean direction with every normalised Fermat curve.

The geometry is therefore especially simple. Every point on a higher Fermat curve is obtained by multiplying the corresponding rational Pythagorean point by the common scale factor

$$\boxed{\lambda_n = \frac{1}{(a^n + b^n)^{1/n}}}$$

which depends only on the coordinates of the original point and the chosen exponent.

Because

$$a^2 + b^2 = 1,$$

with

$$0 < a, b < 1,$$

we have

$$a^n + b^n < 1, \qquad n > 2,$$

and consequently

$$\lambda_n > 1.$$

Thus every projected point lies farther from the origin than the original rational point on the unit circle. This agrees precisely with the visual impression given by Figure 1, where the higher Fermat curves lie outside the circle and approach the boundary of the unit square as the exponent increases.

The projection formula therefore provides the mathematical mechanism underlying the visualisation. The remaining question is no longer geometric but arithmetic: under what circumstances is the scale factor

$$(a^n + b^n)^{1/n}$$

rational, and when must it be irrational?

What the Projection Reveals

The projection formula provides an immediate explanation for the geometry seen in Figure 1.

First, every projected point lies on the same ray as the original Pythagorean point. The projection changes only the distance from the origin; it does not alter the direction. Thus each rational Pythagorean direction generates an entire sequence of associated points, one on every normalised Fermat curve.

Secondly, the projection always moves the point away from the origin. Since

$$0 < a, b < 1,$$

raising either coordinate to a higher power decreases its value. Consequently,

$$a^n + b^n < 1, \qquad n > 2,$$

so the projection factor

$$\lambda_n = \frac{1}{(a^n + b^n)^{1/n}}$$

is always greater than one. This explains why the higher Fermat curves lie outside the unit circle while remaining inside the unit square.

The third observation concerns the arithmetic of the projection. The original point

$$(a, b)$$

is rational because it arises from a Pythagorean triple. Whether the projected point

$$(a', b')$$

is also rational depends entirely upon the denominator

$$(a^n + b^n)^{1/n}.$$

If this quantity is rational, then both projected coordinates remain rational. If it is irrational, then, provided neither numerator is zero, at least one projected coordinate must also be irrational.

The entire arithmetic problem has therefore been reduced to a single quantity. Instead of analysing two coordinates separately, it is enough to determine the nature of the projection factor.

This reduction is one of the principal advantages of the geometric approach. The geometry is completely explicit, and the remaining difficulty is concentrated in a single expression. The rest of the paper investigates this expression for one representative infinite family of Pythagorean triples.

Projection Theorem for the Succession-1 Family

To illustrate how the projection principle may be used, we consider one particularly simple infinite family of Pythagorean triples. Throughout this section the hypotenuse exceeds the longer leg by one,

$$Z = X + 1.$$

For convenience, we introduce the term *succession-1 triples* for this family. The first few examples are

$$(4, 3, 5), \qquad (12, 5, 13), \qquad (24, 7, 25), \qquad (40, 9, 41).$$

Normalising each triple gives the rational points

$$\left(\frac{4}{5}, \frac{3}{5}\right), \quad \left(\frac{12}{13}, \frac{5}{13}\right), \quad \left(\frac{24}{25}, \frac{7}{25}\right), \ldots$$

on the unit circle.

For a general exponent $n > 2$, the corresponding projection factor is

$$\lambda_n = \frac{1}{(a^n + b^n)^{1/n}},$$

where

$$a = \frac{X}{Z}, \qquad b = \frac{Y}{Z}.$$

Since

$$a^2 + b^2 = 1,$$

and

$$0 < a, b < 1,$$

we have

$$a^n + b^n < 1,$$

so every projected point lies beyond the unit circle.

The crucial question is whether

$$(a^n + b^n)^{1/n}$$

can itself be rational.

The projection construction leads directly to the principal result of this article.

Theorem 1 (Projection Theorem)**.** *Let*

$$(X, Y, Z)$$

be any primitive succession-1 Pythagorean triple, so that

$$X^2 + Y^2 = Z^2, \qquad Z = X + 1.$$

Let

$$(a, b) = \left(\frac{X}{Z}, \frac{Y}{Z}\right)$$

be the corresponding rational point on the unit circle, and let

$$(a', b')$$

be the point obtained by extending the ray from the origin through (a, b) *until it meets the normalised Fermat curve*

$$x^n + y^n = 1, \qquad n > 2.$$

Then, for every integer $n > 2$*,*

$$(a', b') = \left(\frac{a}{(a^n + b^n)^{1/n}}, \frac{b}{(a^n + b^n)^{1/n}} \right),$$

and both coordinates a' *and* b' *are irrational.*

Consequently, every primitive succession-1 Pythagorean direction meets every higher normalised Fermat curve only at irrational points.

Proof. For succession-1 triples we have

$$Z = X + 1.$$

Since $n > 2$,

$$X^n + Y^n < Z^n = (X + 1)^n,$$

so

$$X < (X^n + Y^n)^{1/n} < X + 1.$$

Hence $X^n + Y^n$ is not a perfect n-th power, and therefore

$$(X^n + Y^n)^{1/n}$$

is irrational [3].

Normalising by Z gives

$$(a, b) = \left(\frac{X}{Z}, \frac{Y}{Z} \right),$$

and the Projection Principle gives

$$a' = \frac{a}{(a^n + b^n)^{1/n}}, \qquad b' = \frac{b}{(a^n + b^n)^{1/n}}.$$

Since a and b are non-zero rationals while the common denominator is irrational, both projected coordinates are irrational. □

The theorem demonstrates how the geometric projection converts an infinite family of rational Pythagorean points into irrational points on every higher normalised Fermat curve. It illustrates the central theme of this article: the visualisation reduces the geometric problem to the arithmetic of a single projection factor.

A Geometric Summary

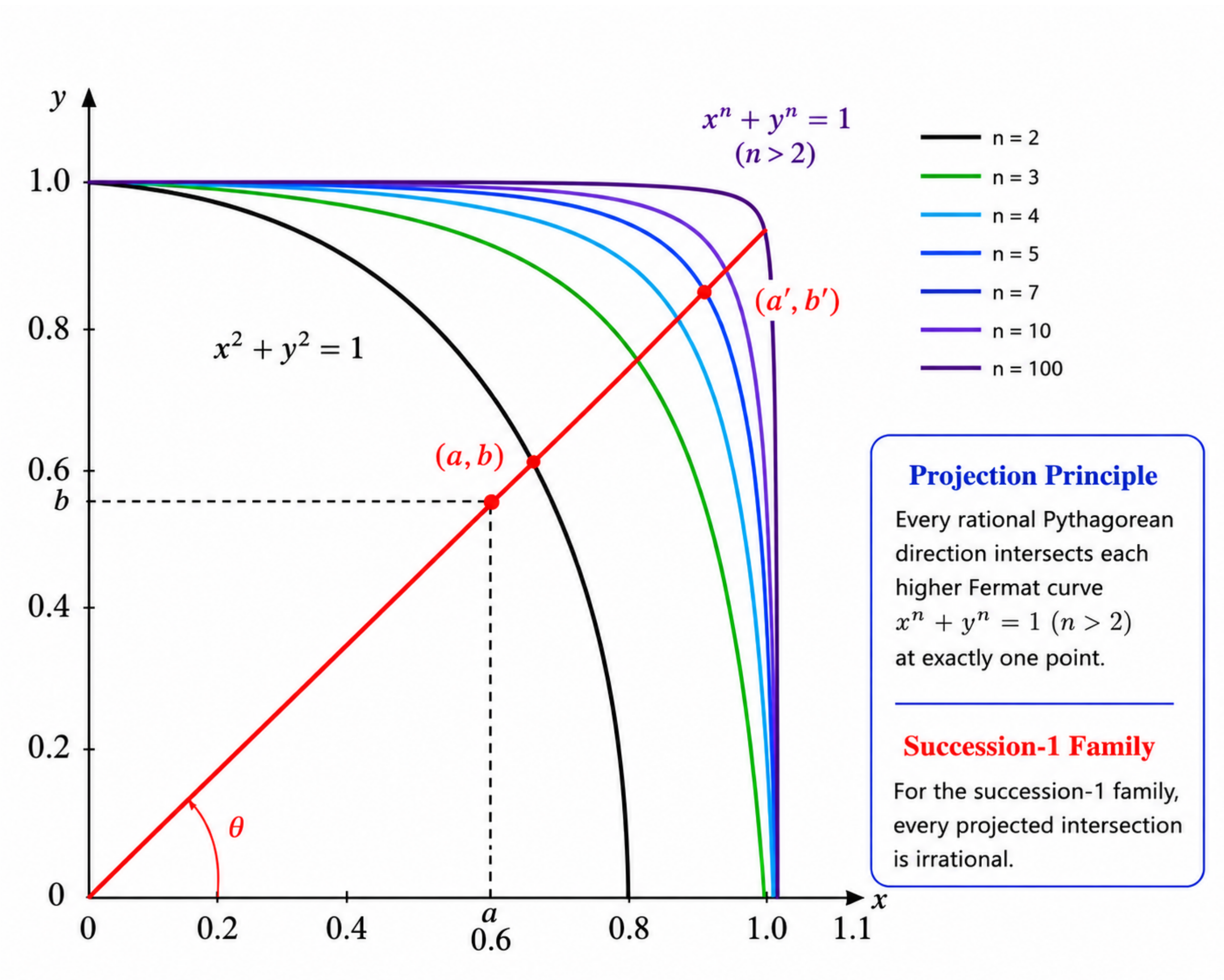

Figure 3: **Summary of the Projection Principle.** Each rational Pythagorean direction intersects every higher normalised Fermat curve $x^n + y^n = 1$ $(n > 2)$ at a unique projected point. The Projection Principle expresses this point as a simple scaling of the corresponding rational point on the unit circle. For the representative succession-1 family considered in this article, Theorem 1 shows that every projected intersection is irrational.

Discussion

The objective of this article has not been to provide an alternative proof of Fermat's Last Theorem, but rather to develop a geometric framework through which one aspect of the theorem may be viewed.

Beginning with the rational points on the unit circle arising from Pythagorean triples, we introduced a projection that associates each rational direction with a unique point on every higher normalised Fermat curve. This construction leads naturally to the Projection Principle, reducing the geometry to the arithmetic of a single scale factor.

For the representative infinite family considered here, that scale factor is shown to be irrational for every exponent $n > 2$. It follows that every projected point on the higher Fermat curves is irrational, excluding corresponding non-trivial solutions of Fermat's equation within this family.

Returning to Figure 1, we now recognise the red crosses as more than rational points on the unit circle. Each determines a ray whose intersections with the higher Fermat curves are governed by the Projection Principle. For the representative family studied here, every one of these projected intersections is necessarily irrational.

Although only the succession-1 family has been considered here, the projection framework itself is more general. Other infinite families of Pythagorean triples, including the succession-2 and succession-8 families, can be analysed using the same geometric construction [4].

The visualisation therefore complements the classical theory by making one aspect of Fermat's Last Theorem directly visible. Rather than replacing the deep arithmetic underlying the theorem, it provides an elementary geometric interpretation that links the familiar Pythagorean equation to the family of higher-order Fermat curves and offers a geometric perspective on the transition from the Pythagorean equation to Fermat's higher-power equations.

References

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[2] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford University Press, 2008.

[3] Ivan Niven, *Numbers Rational and Irrational*, The Mathematical Association of America, 1961.

[4] Brian Scannell, *Visualising Fermat's Last Theorem with Proofs Within Infinite Families of Pythagorean Triples*, viXra:2601.0024, January 2026.