The Complex Lorentz Factor and Its Hidden Cartesian Grid

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Abstract

The Lorentz factor

$$\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}},$$

where $\beta = v/c$, is normally considered for real $|\beta| < 1$. Allowing its argument to range over the complex plane gives

$$\Gamma(z) = \frac{1}{\sqrt{1-z^2}},$$

whose real and imaginary parts generate a striking system of level curves organized around the branch points $z = \pm 1$. The Cauchy–Riemann equations explain why the regular curves from the two families intersect orthogonally. More surprisingly, eliminating the square root shows that each nonzero contour lies on a degree-eight algebraic locus. This apparent complexity is deceptive. Solving instead for z gives

$$z = \pm\sqrt{1-\Gamma^{-2}},$$

revealing that the apparently complicated contours in the z-plane are locally the preimages of vertical and horizontal straight lines in the Γ-plane. Thus a familiar real function from special relativity provides an elementary example in which complicated algebraic contours conceal a simple conformal geometry.

1. Leaving the real axis

One of the most familiar expressions in special relativity is the Lorentz factor

$$\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}. \tag{1}$$

For its usual physical interpretation, β is real and satisfies

$$-1 < \beta < 1.$$

The function is then real and positive, with $\gamma(\beta) \to \infty$ as β approaches either 1 or -1; see, for example, [1].

Complex velocity transformations already occur in mathematical physics [5]. Our purpose is much more elementary: no physical interpretation of a complex velocity is assumed here. We simply ask what geometry appears if the argument of (1) is allowed to leave the real axis.

Let

$$z = x + iy$$

and consider the analytic continuation

$$\Gamma(z) = \frac{1}{\sqrt{1-z^2}}. \tag{2}$$

We use Γ to distinguish the complex function from the usual real Lorentz factor.

The points $z = -1$ and $z = 1$ are immediately distinguished. On the real axis these are the familiar singular values $\beta = \pm 1$. In the complex plane they become square-root branch points. For example, near $z = 1$,

$$1 - z^2 = -(z-1)(z+1) \sim -2(z-1),$$

and hence

$$\Gamma(z) \sim \frac{1}{\sqrt{-2(z-1)}}. \tag{3}$$

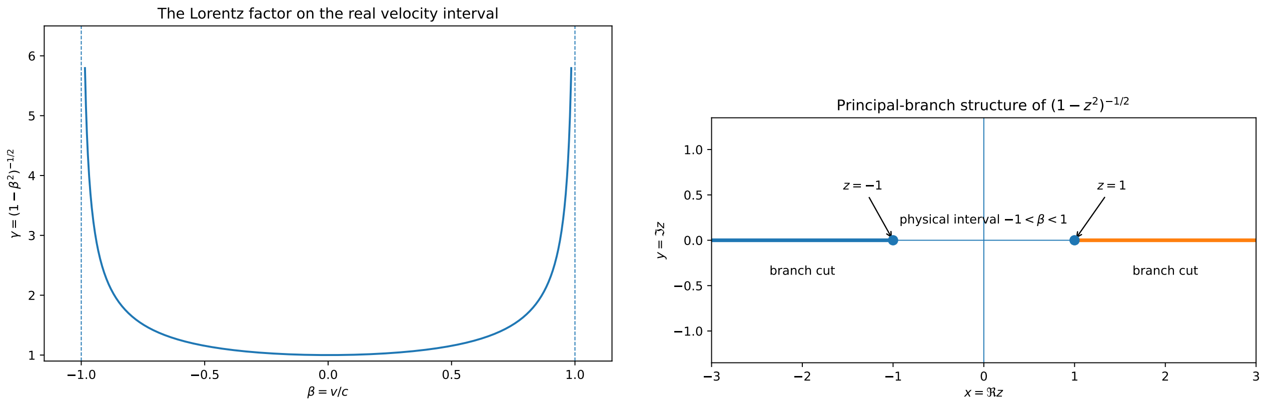
Similarly,

$$\Gamma(z) \sim \frac{1}{\sqrt{2(z+1)}} \tag{4}$$

near $z=-1$, up to the phase determined by the chosen branch.
For definiteness we use the principal square root. Its natural domain is

$$D = \mathbb{C} \setminus \big((-\infty,-1] \cup [1,\infty)\big). \tag{5}$$

Thus the ordinary physical interval $-1<\beta<1$ lies between the two branch points of the complex continuation.



(a) The ordinary real Lorentz factor. **(b)** Branch points and principal branch cuts.

Figure 1: The real Lorentz factor and its complex continuation. The singular values $\beta=\pm1$ become square-root branch points in the complex z-plane.

The continuation (2) is an elementary complex function. Its geometry, however, is less elementary in appearance.

2. Two families of contours

Write

$$\Gamma(x+iy) = u(x,y) + iw(x,y), \tag{6}$$

where $u = \Re\Gamma$ and $w = \Im\Gamma$.
Since

$$1-(x+iy)^2 = 1-x^2+y^2-2ixy,$$

it is convenient to put

$$A = 1-x^2+y^2 \tag{7}$$

and

$$S = A^2 + 4x^2y^2. \tag{8}$$

The latter has the useful factorization

$$S = \big((x-1)^2+y^2\big)\big((x+1)^2+y^2\big), \tag{9}$$

so S vanishes precisely at the two branch points.
Squaring (6),

$$(u+iw)^2 = \frac{1}{1-z^2} = \frac{A+2ixy}{S}.$$

Therefore

$$u^2 - w^2 = \frac{A}{S}. \tag{10}$$

On the other hand,

$$u^2 + w^2 = |\Gamma|^2 = \frac{1}{\sqrt{S}}. \tag{11}$$

Combining these equations gives

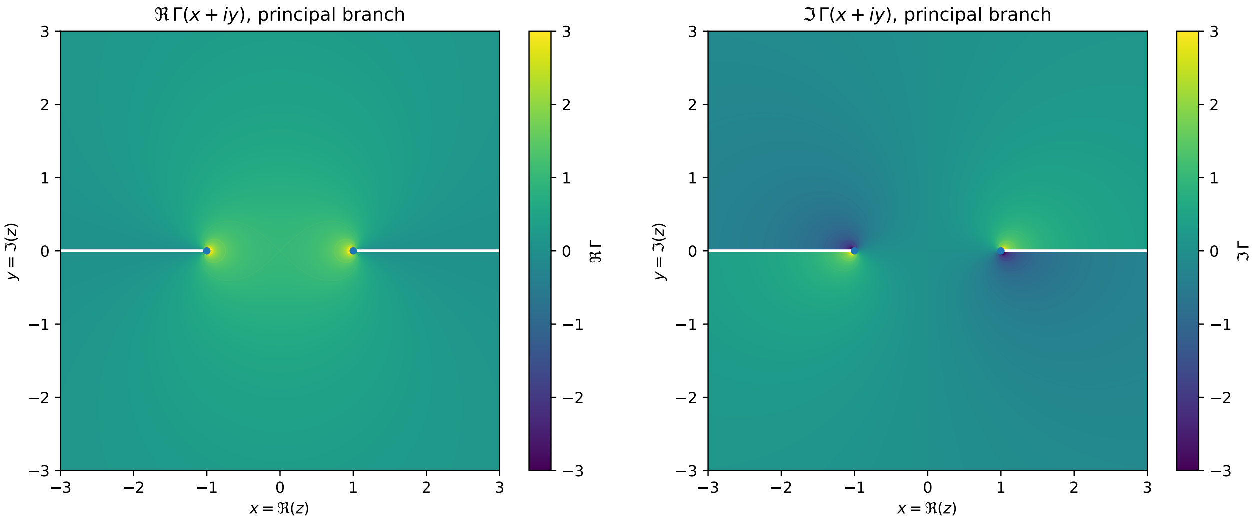
$$u^2 = \frac{\sqrt{S}+A}{2S}, \tag{12}$$

and

$$w^2 = \frac{\sqrt{S} - A}{2S}. \tag{13}$$

On the principal branch,

$$u = \Re\Gamma > 0. \tag{14}$$



(a) $u = \Re\Gamma$. **(b)** $w = \Im\Gamma$.

Figure 2: Real and imaginary parts of $\Gamma(z) = (1 - z^2)^{-1/2}$ on the principal branch. The singularities occur at the branch points $z = \pm 1$. The colour scale is truncated near the branch points to make the surrounding structure visible.

The complementary structure of the two pictures is a standard consequence of analyticity. Away from the branch cuts and branch points, u and w satisfy the Cauchy–Riemann equations

$$u_x = w_y, \qquad u_y = -w_x. \tag{15}$$

They are therefore harmonic conjugates:

$$\nabla^2 u = \nabla^2 w = 0. \tag{16}$$

Moreover,

$$\nabla u \cdot \nabla w = u_x w_x + u_y w_y = 0. \tag{17}$$

Thus whenever the gradients are nonzero, the level curves of u and w intersect orthogonally. This is the familiar geometry of harmonic conjugates [2, 3].

For the present function,

$$\Gamma'(z) = \frac{z}{(1 - z^2)^{3/2}}, \tag{18}$$

so, apart from the branch singularities, the only finite critical point is

$$z = 0. \tag{19}$$

None of this orthogonality is peculiar to the Lorentz factor: it is a standard property of the real and imaginary parts of an analytic function. What is particular to (2) is the shape of the resulting curves.

So what curves are we actually looking at?

3. An unexpectedly complicated answer

Consider first a nonzero real-part contour

$$u = c.$$

From (12),

$$c^2 = \frac{\sqrt{S} + A}{2S},$$

and therefore

$$2c^2 S - A = \sqrt{S}. \tag{20}$$

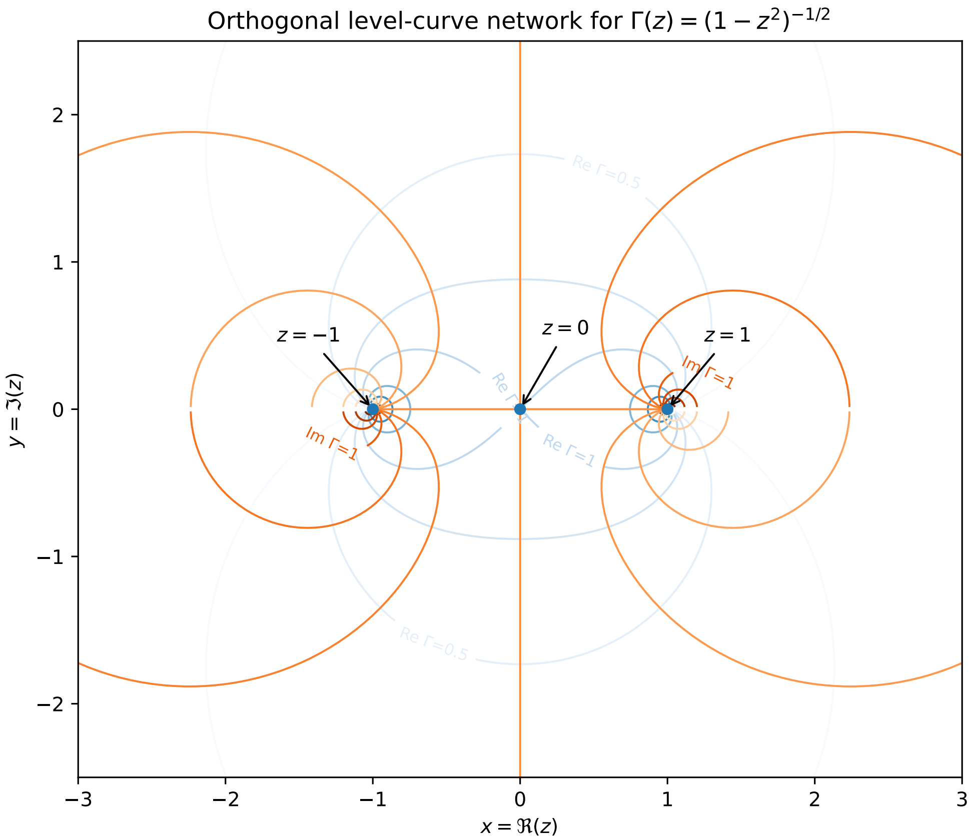


Figure 3: Level curves of $u = \Re\Gamma$ and $w = \Im\Gamma$. Away from the critical point $z = 0$ and the branch singularities at $z = \pm 1$, the two families meet orthogonally, as required by the Cauchy–Riemann equations.

Squaring gives

$$\boxed{(2c^2 S - A)^2 = S.} \tag{21}$$

Recall that $A = 1 - x^2 + y^2$ and

$$S = (1 - x^2 + y^2)^2 + 4x^2y^2.$$

The polynomial equation (21) has total degree eight whenever $c \neq 0$. Indeed, its highest-degree contribution is

$$4c^4(x^2 + y^2)^4. \tag{22}$$

The imaginary-part contours behave similarly. If

$$w = d, \qquad d \neq 0,$$

then (13) gives

$$2d^2 S + A = \sqrt{S}, \tag{23}$$

and consequently

$$\boxed{(2d^2 S + A)^2 = S.} \tag{24}$$

Again the resulting polynomial has total degree eight, with highest-degree contribution

$$4d^4(x^2 + y^2)^4. \tag{25}$$

The squaring in (21) and (24) requires a qualification. The required portions of these algebraic loci must also satisfy

$$2c^2 S - A \geq 0 \tag{26}$$

and

$$2d^2 S + A \geq 0, \tag{27}$$

respectively, together with the appropriate sign and branch conditions. Thus it is more precise to say that each nonzero contour is a *branch-selected subset of a degree-eight algebraic locus.*

The zero levels are exceptional. On the principal branch $u > 0$, so no interior contour $u = 0$ occurs. For $w = 0$, equations (8) and (13) yield

$$xy = 0$$

subject to $A \geq 0$. Hence the zero imaginary contour consists of the imaginary axis together with the real segment

$$-1 < x < 1.$$

They meet at the critical point $z = 0$.

The degree-eight equations answer our question, but not very satisfactorily. Figure 3 now seems to consist of two interlocking families of rather complicated algebraic curves.

That complexity largely disappears when the same curves are viewed in the Γ-plane.

4. The hidden Cartesian grid

Instead of eliminating Γ, solve (2) for z. We have

$$\Gamma^{-2} = 1 - z^2,$$

and therefore

$$z^2 = 1 - \Gamma^{-2}.$$

Thus, locally,

$$\boxed{z = \pm\sqrt{1 - \Gamma^{-2}}.} \tag{28}$$

The sign cannot be discarded because

$$\Gamma(z) = \Gamma(-z). \tag{29}$$

Consequently the inverse is not globally single-valued. On any region where one branch of (28) can be chosen analytically, however, the geometry becomes almost trivial.

A contour $u = c$ means simply

$$\Re\Gamma = c.$$

In the Γ-plane this is a vertical straight line. Writing

$$\Gamma = c + it$$

and substituting into (28) gives the local parametrization

$$\boxed{z(t) = \pm\sqrt{1 - \frac{1}{(c + it)^2}}.} \tag{30}$$

Similarly, $w = d$ is simply the horizontal line

$$\Im\Gamma = d.$$

Writing $\Gamma = t + id$ gives

$$\boxed{z(t) = \pm\sqrt{1 - \frac{1}{(t + id)^2}}.} \tag{31}$$

Thus the apparently complicated degree-eight contours are the inverse images of the simplest possible curves: straight coordinate lines.

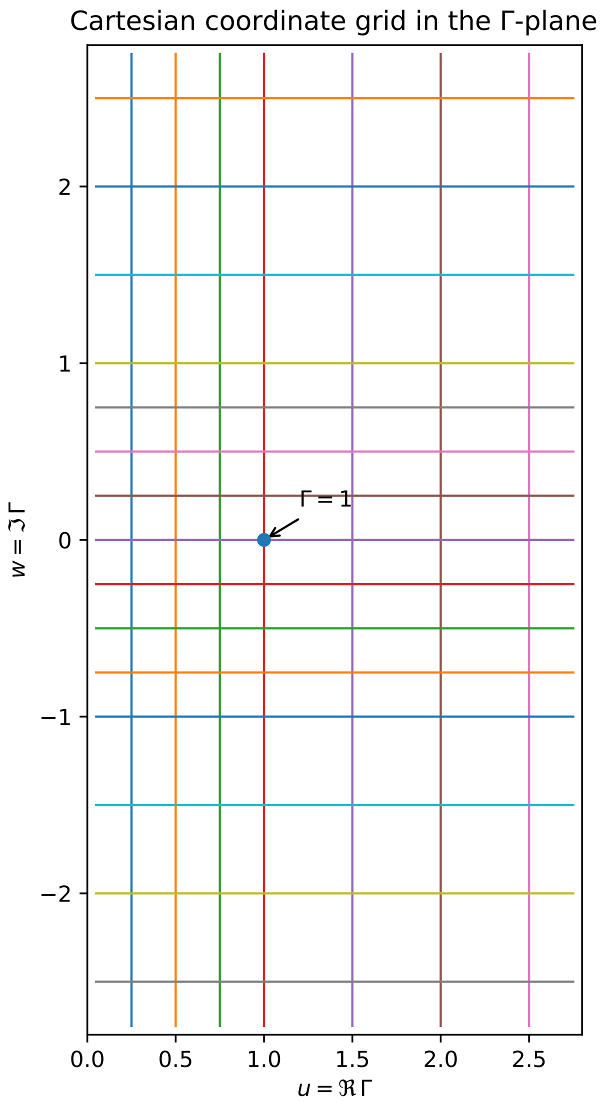
This gives a more revealing description of the construction than the degree-eight equations. The contour network of the complex Lorentz factor is locally a *Cartesian grid viewed through an inverse conformal map.*

To make this precise, let

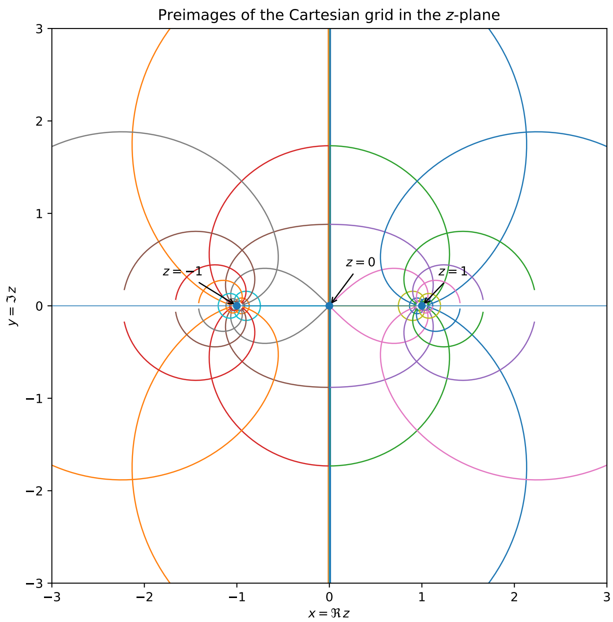
$$F(\Gamma) = \sqrt{1 - \Gamma^{-2}} \tag{32}$$

denote a chosen analytic branch on some suitable domain in the Γ-plane. Wherever $F'(\Gamma) \neq 0$, the map is conformal. Since vertical and horizontal Cartesian lines meet at right angles, their images under F do also.

This gives the same orthogonality seen earlier from the Cauchy–Riemann equations, but from the opposite direction. In the z-plane we begin with apparently complicated contours and discover that they are orthogonal



(a) Cartesian grid in the Γ-plane.



(b) Their preimages in the z-plane.

Figure 4: The hidden Cartesian grid. Vertical lines $u = \Re\Gamma = \text{constant}$ and horizontal lines $w = \Im\Gamma = \text{constant}$ form an ordinary Cartesian coordinate grid in the right half of the Γ-plane. Their local preimages under $\Gamma(z) = (1 - z^2)^{-1/2}$ are the apparently complicated curvilinear level sets shown in the z-plane; equivalently, these curves are obtained locally by applying the inverse branch $z = \pm\sqrt{1 - \Gamma^{-2}}$ to the grid lines. Away from critical and branch points the map is locally conformal, so the orthogonality of the Cartesian grid is preserved.

because Γ is analytic. In the Γ-plane we begin with an ordinary rectangular grid and watch an inverse analytic map bend it into those same contours.

The two descriptions are equivalent:

$$\boxed{\begin{array}{c}\text{degree-eight algebraic loci in the } z\text{-plane}\\ \updownarrow \\ \text{straight Cartesian coordinate lines in the } \Gamma\text{-plane.}\end{array}} \tag{33}$$

This contrast is the main geometric feature of the construction.

5. What happens at the centre?

There is one point at which the regular-grid picture degenerates. Around $z = 0$, the binomial expansion gives

$$\Gamma(z) = (1 - z^2)^{-1/2} = 1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \cdots . \tag{34}$$

To lowest nonconstant order,

$$\Gamma(x + iy) \approx 1 + \frac{1}{2}(x + iy)^2.$$

Consequently,

$$u - 1 \approx \frac{1}{2}(x^2 - y^2), \tag{35}$$

and

$$w \approx xy. \tag{36}$$

Thus near the origin the two contour families are approximately

$$x^2 - y^2 = \text{constant} \tag{37}$$

and

$$xy = \text{constant}. \tag{38}$$

These are two orthogonal families of rectangular hyperbolas.

The exceptional geometry follows from the fact that $\Gamma'(0) = 0$. Indeed, (34) shows that the leading map near the origin is quadratic:

$$\Gamma - 1 \sim \frac{1}{2}z^2. \tag{39}$$

Near this point, the preimages of the ordinary right-angle grid in the Γ-plane are therefore governed to leading order by the locally two-to-one quadratic map $\Gamma - 1 \sim z^2/2$.

The branch points $z = \pm 1$ exhibit a different local geometry. From (3),

$$|\Gamma(z)| \sim \frac{1}{\sqrt{2|z - 1|}} \tag{40}$$

near $z = 1$, with an analogous expression near $z = -1$. The half-angle dependence of the phase is characteristic of a square-root branch point.

The contour network therefore has three naturally distinguished real points:

$$-1, \qquad 0, \qquad 1.$$

The outer two are branch points inherited from the light-speed singularities of the real Lorentz factor; the central point is the critical point at which the otherwise regular orthogonal coordinate structure degenerates.

6. A familiar complex-analytic setting

The function in (2) is, of course, not new to complex analysis. In particular,

$$\frac{d}{dz} \arcsin z = \frac{1}{\sqrt{1 - z^2}},$$

so

$$\Gamma(z) = (\arcsin z)'. \tag{41}$$

Likewise, appropriate branch choices give

$$\Gamma(z) = \pm \frac{i}{\sqrt{z^2 - 1}}, \tag{42}$$

and hence locally

$$\Gamma(z) = \pm i \, (\operatorname{arccosh} z)'. \tag{43}$$

Functions involving square roots of $z^2 - 1$ occur naturally in classical conformal mapping and complex-potential theory [3, 4, 6]. In those settings, too, real and imaginary parts of analytic functions provide harmonic conjugate coordinates and hence orthogonal families of curves.

The purpose here is not to assign such a physical interpretation to Γ. Rather, these classical connections place the contour geometry of the complex Lorentz factor within a familiar complex-analytic setting. What is distinctive about the present example is its starting point: one of the most elementary and recognizable real functions of special relativity.

7. Back to the real Lorentz factor

We began with

$$\gamma(\beta) = \frac{1}{\sqrt{1 - \beta^2}},$$

a real function ordinarily considered only on $-1 < \beta < 1$. Allowing the argument to enter the complex plane does not require interpreting complex values of v/c as physical velocities. It simply exposes the complex geometry of the function already present in (1).

The singular values $\beta = \pm 1$ become square-root branch points. The real and imaginary components of the continuation are harmonic conjugates whose regular level curves meet orthogonally. Algebraically, each nonzero contour lies on a branch-selected degree-eight locus.

That algebraic complexity initially appears to be the main feature of the pictures.

It is not.

Writing $\Gamma = u + iw$ reveals that the same curves become nothing more than

$$u = \text{constant}, \qquad w = \text{constant}$$

in the Γ-plane. The degree-eight contour network consists locally of the preimages of the vertical and horizontal lines of a Cartesian grid in the Γ-plane. Equivalently, on a chosen inverse branch these curves are generated by

$$z = \pm\sqrt{1 - \Gamma^{-2}}.$$

A complicated algebraic picture is therefore hiding a simple conformal one.

This provides a compact example of a recurring theme in complex analysis: curves that look formidable in one coordinate system can become elementary after the right analytic change of variable. Here that change of variable happens to be supplied by one of the most familiar formulas in special relativity. No physical interpretation of complex v/c is assumed here. Nevertheless, mathematical extensions introduced formally have sometimes acquired physical meaning only later. It remains conceivable that $\Gamma(z)$, or its complex structure, could have an as-yet-unknown physical significance; at present, this is purely speculative.

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