From Piled Cubes to the Geometry of Apéry's Constant

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Abstract

Passare's geometric interpretation of the Basel problem represents $\zeta(2)$ as the area of a logarithmic region. Motivated by this construction, we develop a natural three-dimensional analogue beginning with piled cubes of side lengths $1/n$.

Replacing each cube by an equal-volume exponential layer yields a logarithmic surface whose enclosed volume is Apéry's constant, $\zeta(3)$. The construction extends naturally to higher dimensions, giving logarithmic hypervolume representations for the sequence $\zeta(m)$.

Returning to Passare's original logarithmic region, we show that its first moment is $\zeta(3)$ and that its centroid is determined by the ratio $\zeta(3)/\zeta(2)$, providing a second geometric interpretation of Apéry's constant. A simple paper model illustrates this interpretation.

1 Introduction

The Basel identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

is one of the classical results of mathematics. Although Euler's original proof was analytic, many later derivations have revealed unexpected geometric interpretations of this remarkable identity.

Among the most elegant is the construction of Passare [1], who showed that the region beneath the logarithmic curve

$$y = -\log(1 - e^{-x})$$

has area exactly $\zeta(2)$. This naturally raises the question of whether the same geometric idea extends beyond area to provide an equally natural geometric interpretation of Apéry's constant,

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

In this paper we answer this question by beginning with piled cubes of side lengths $1/n$. Replacing each cube by an equal-volume exponential layer leads to a logarithmic surface whose enclosed volume is precisely $\zeta(3)$. The construction extends naturally to higher dimensions, yielding logarithmic hypervolume representations for the sequence $\zeta(m)$.

Returning to Passare's original logarithmic region, we then show that the same geometry already encodes Apéry's constant through its first moment and the centroid of the region. Thus Passare's geometric interpretation of the Basel problem leads naturally to the geometry of Apéry's constant.

The piled-cube construction underlying this article was first developed by the author in 2016 [2]. The present paper revisits that construction, gives it a more systematic analytic treatment, extends the logarithmic-surface viewpoint to higher dimensions, and develops the additional moment and centroid interpretations considered below.

2 Passare's Curve and the Basel Problem

Passare's elegant geometric interpretation of the Basel problem [1] provides the starting point for the construction developed in this paper.

The classical identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

admits a geometric interpretation through the logarithmic curve $y = -\log(1 - e^{-x})$, whose enclosed area is exactly $\zeta(2)$.

Figure 1 shows Passare's original construction.

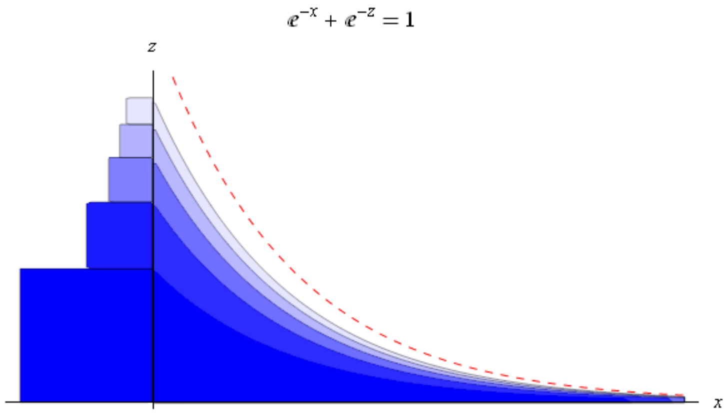

Figure 1: Passare's geometric interpretation of the Basel problem [1]. The region beneath the logarithmic curve $y = -\log(1 - e^{-x})$ has area $\zeta(2)$.

An equivalent viewpoint, more suited to higher-dimensional generalisation, begins with squares of side lengths

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots,$$

whose total area is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Associate the nth square with the exponential layer

$$f_n(x) = \frac{e^{-nx}}{n}, \qquad x \geq 0.$$

Its area is

$$\int_0^\infty \frac{e^{-nx}}{n}\, dx = \frac{1}{n^2},$$

so each exponential layer has exactly the same area as the corresponding square. Matching colours in Figure 2 indicate these equal-area correspondences.

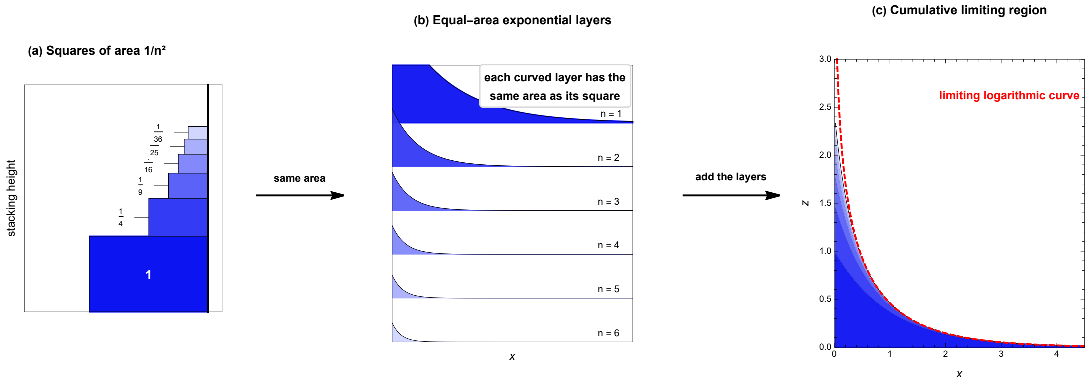

Figure 2: An equal-area interpretation of Passare's logarithmic region. (a) Squares of area $1/n^2$. (b) Equal-area exponential layers. (c) Their cumulative upper boundaries converge to Passare's logarithmic curve. Matching colours denote equal-area contributions.

After the first N layers,

$$S_N(x) = \sum_{n=1}^{N} \frac{e^{-nx}}{n},$$

and the Mercator expansion,

$$-\log(1-u) = \sum_{n=1}^{\infty} \frac{u^n}{n}, \qquad |u| < 1,$$

with $u = e^{-x}$ gives

$$S_N(x) \longrightarrow -\log(1-e^{-x}), \qquad N \to \infty.$$

Thus Passare's logarithmic curve is the limiting upper boundary of the equal-area exponential layers.

Proposition 1. *The area beneath Passare's logarithmic curve is*

$$\int_0^\infty -\log(1-e^{-x})\, dx = \zeta(2).$$

Proof. For $x > 0$,

$$-\log(1-e^{-x}) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{n}.$$

Since every term is non-negative, Tonelli's theorem permits the order of summation and integration to be interchanged. Hence

$$\begin{aligned} \int_0^{\infty} -\log(1-e^{-x})\,dx &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} e^{-nx}\,dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \zeta(2). \end{aligned}$$

□

This equal-area construction provides the geometric template for the three-dimensional logarithmic surface developed in the next section.

3 Piled Cubes and a Logarithmic Surface for $\zeta(3)$

The equal-area construction of Section 2 admits a natural three-dimensional analogue. We replace squares by cubes of side lengths

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots,$$

whose total volume is $\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$.

Each cube is replaced by an equal-volume exponential layer, whose cumulative upper boundaries converge to the logarithmic surface shown schematically in Figure 3.

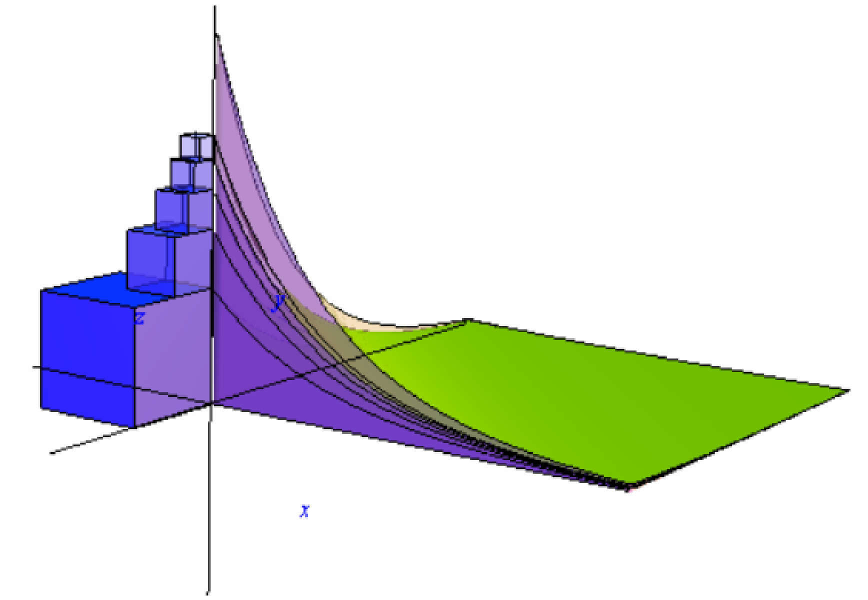

Figure 3: From piled cubes to a logarithmic surface. Each cube of volume $1/n^3$ is replaced by an equal-volume exponential layer. Matching colours denote equal-volume contributions, and the cumulative upper boundaries converge to the logarithmic surface.

Associate the nth cube with the exponential layer

$$g_n(x,y) = \frac{e^{-n(x+y)}}{n}, \qquad x, y \geq 0.$$

Its volume is

$$\int_0^\infty \int_0^\infty \frac{e^{-n(x+y)}}{n}\, dx\, dy = \frac{1}{n}\left(\int_0^\infty e^{-nx}\, dx\right)^2$$
$$= \frac{1}{n^3},$$

which is exactly the volume of the corresponding cube.

A natural first idea is to rotate Passare's logarithmic curve about an axis. Although geometrically appealing, the resulting solid has volume $\pi\zeta(3)/2$ rather than $\zeta(3)$ and therefore does not provide the required construction.
Instead, the required construction is obtained by allowing the exponential decay to depend only on the combined coordinate $x + y$.

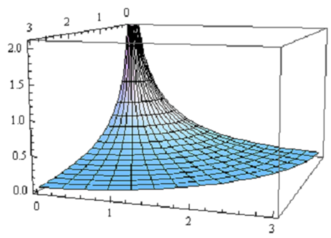

Figure 4: A surface of revolution obtained from Passare's logarithmic curve. Its volume is $\pi\zeta(3)/2$, not $\zeta(3)$.

Using the Mercator expansion,

$$-\log\left(1 - e^{-(x+y)}\right) = \sum_{n=1}^{\infty} \frac{e^{-n(x+y)}}{n},$$

the cumulative upper boundary after the first N layers is

$$T_N(x,y) = \sum_{n=1}^{N} \frac{e^{-n(x+y)}}{n},$$

so that

$$T_N(x,y) \longrightarrow -\log\left(1 - e^{-(x+y)}\right), \qquad N \to \infty.$$

Thus the logarithmic surface $z = -\log\left(1 - e^{-(x+y)}\right)$ is the limiting upper boundary of the equal-volume exponential layers.

Proposition 2. *The region beneath the logarithmic surface*

$$z = -\log\left(1 - e^{-(x+y)}\right), \qquad x, y \geq 0,$$

has volume

$$\int_0^\infty \int_0^\infty -\log\left(1 - e^{-(x+y)}\right)\, dx\, dy = \zeta(3).$$

Proof. For $x, y > 0$,

$$-\log\bigl(1 - e^{-(x+y)}\bigr) = \sum_{n=1}^{\infty} \frac{e^{-n(x+y)}}{n}.$$

Since every term is non-negative, Tonelli's theorem permits the order of summation and integration to be interchanged. Hence

$$\begin{aligned}
&\int_0^\infty \int_0^\infty -\log\bigl(1 - e^{-(x+y)}\bigr)\ dx\, dy \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty e^{-nx}\, dx \int_0^\infty e^{-ny}\, dy \\
&= \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3).
\end{aligned}$$

$\square$

Thus Passare's logarithmic curve extends naturally to a logarithmic surface whose enclosed volume is $\zeta(3)$.

3.1 Higher-Dimensional Extension

The logarithmic surface is the first non-trivial member of a natural higher-dimensional family.

Each additional dimension contributes an independent exponential integral, $\int_0^\infty e^{-nx}\, dx = \frac{1}{n}$, and hence one further factor of $1/n$.

Proposition 3. *For every integer $k \geq 1$,*

$$\int_{[0,\infty)^k} -\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr)\ dx_1 \cdots dx_k = \zeta(k+1).$$

Proof. For $x_1, \ldots, x_k > 0$,

$$-\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr) = \sum_{n=1}^{\infty} \frac{e^{-n(x_1+\cdots+x_k)}}{n}.$$

Since every term is non-negative, Tonelli's theorem gives

$$\begin{aligned}
&\int_{[0,\infty)^k} -\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr)\ dx_1 \cdots dx_k \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \prod_{j=1}^{k} \int_0^\infty e^{-nx_j}\, dx_j \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n}\right)^k = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \zeta(k+1).
\end{aligned}$$

$\square$

For $k = 1$ this recovers Passare's logarithmic region with area $\zeta(2)$, while $k = 2$ gives the logarithmic surface above with volume $\zeta(3)$. Further dimensions yield logarithmic hypervolume representations for $\zeta(4), \zeta(5)$ and, more generally, $\zeta(m)$.

We now return to Passare's original logarithmic region, where Apéry's constant appears again through its moments and centroid.

4 Moments and the Centroid Interpretation of $\zeta(3)$

Passare's logarithmic region contains more geometric information than its area alone. In particular, its first moment about the y-axis yields a second geometric interpretation of Apéry's constant.

Let

$$A = \left\{(x, y) : x \geq 0,\ 0 \leq y \leq -\log(1 - e^{-x})\right\}.$$

For a lamina of uniform density occupying A, the first moment about the y-axis is

$$\int_A x\, dA.$$

Proposition 4. *The first moment of Passare's logarithmic region satisfies*

$$\int_A x\, dA = \zeta(3).$$

Consequently,

$$\bar{x} = \frac{\zeta(3)}{\zeta(2)},$$

or equivalently,

$$\zeta(3) = \bar{x}\, \zeta(2).$$

Proof. Using the Mercator expansion,

$$-\log(1 - e^{-x}) = \sum_{n=1}^{\infty} \frac{e^{-nx}}{n},$$

Tonelli's theorem gives

$$\begin{aligned}\int_A x\, dA &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} x e^{-nx}\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3).\end{aligned}$$

Since $\int_A dA = \zeta(2)$, the centroid therefore satisfies

$$\bar{x} = \frac{\displaystyle\int_A x\, dA}{\displaystyle\int_A dA} = \frac{\zeta(3)}{\zeta(2)},$$

or equivalently,

$$\zeta(3) = \bar{x}\,\zeta(2).$$

$\square$

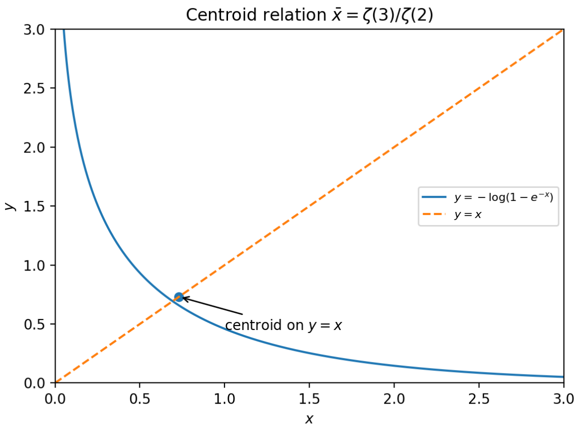

Figure 5: The centroid of Passare's logarithmic region. Since the area is $\zeta(2)$ and the first moment is $\zeta(3)$, the centroid satisfies $\bar{x} = \zeta(3)/\zeta(2)$.

Thus Apéry's constant admits two complementary geometric interpretations: as the volume beneath the logarithmic surface of Section 3, and as the first moment, or equivalently the centroid, of Passare's original logarithmic region.

More generally,

$$\int_A x^m\, dA = m!\,\zeta(m+2),$$

so successive moments generate the higher values $\zeta(4), \zeta(5), \ldots$.

Appendix A describes a simple paper model illustrating the centroid interpretation experimentally.

5 Concluding Remarks

Beginning with Passare's geometric interpretation of the Basel problem, we have shown how the same logarithmic geometry extends naturally from area to volume. Replacing piled cubes by equal-volume exponential layers leads to a logarithmic surface whose enclosed volume is Apéry's constant, while the construction extends naturally to higher-dimensional logarithmic hypervolumes.

Returning to Passare's logarithmic region reveals a second geometric interpretation of Apéry's constant. Its first moment is $\zeta(3)$ and its centroid satisfies

$$\zeta(3) = \bar{x}\,\zeta(2),$$

showing that the geometry of the original two-dimensional region already encodes Apéry's constant.

Appendix A illustrates this interpretation with a simple paper model, providing an exploratory physical demonstration of the centroid construction.

A A Paper Model Illustrating the Centroid Interpretation

Section 4 showed that the centroid of Passare's logarithmic region satisfies $\bar{x} = \zeta(3)/\zeta(2)$, or equivalently $\zeta(3) = \bar{x}\,\zeta(2)$.

This appendix describes a simple paper model illustrating this geometric interpretation. It is intended as an exploratory demonstration rather than a precision measurement.

The logarithmic region

$$A = \{(x, y) : x \geq 0,\ 0 \leq y \leq -\log(1 - e^{-x})\}$$

was plotted using equal scales on both axes and printed on thick paper. Since the theoretical region is unbounded, the practical model was truncated to

$$0 \leq x \leq 4, \qquad 0 \leq y \leq 4.$$

The printed region was cut out and suspended from several different points using a pin and a plumb line. Each suspension determines a vertical line through the centroid. The corresponding lines were transferred to an uncut copy of the graph, and their intersections with the symmetry line $y = x$ provided three independent estimates of the centroid coordinate.

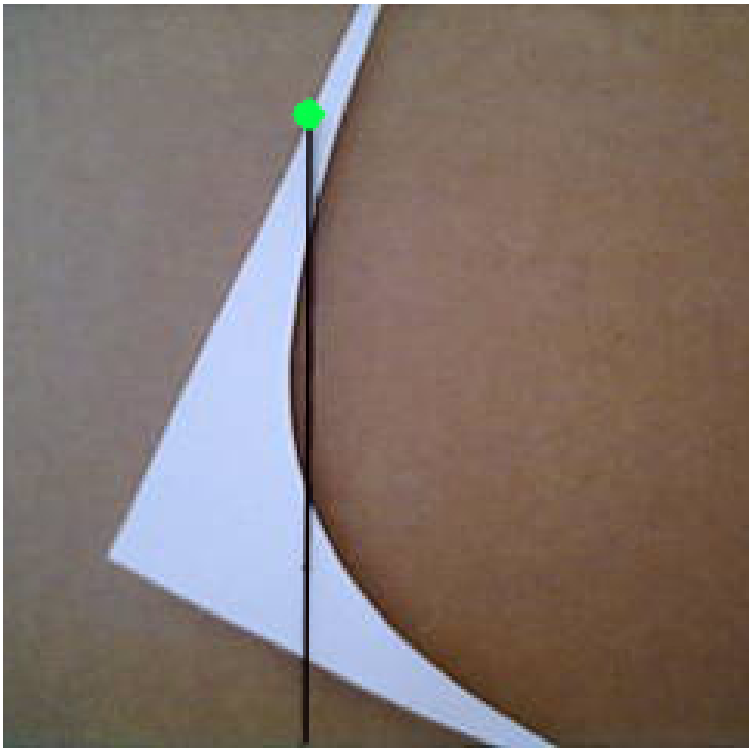

Figure 6: Determining the centroid using a pin and plumb line.

The resulting estimates are shown in Figure 7, giving $\bar{x} \approx 0.73 \pm 0.02$.

Using $\zeta(3) = \bar{x}\,\zeta(2)$ and $\zeta(2) = \pi^2/6$ gives the experimental estimate

$$\zeta(3) \approx 1.20 \pm 0.03,$$

to be compared with $\zeta(3) = 1.2020569\ldots$.

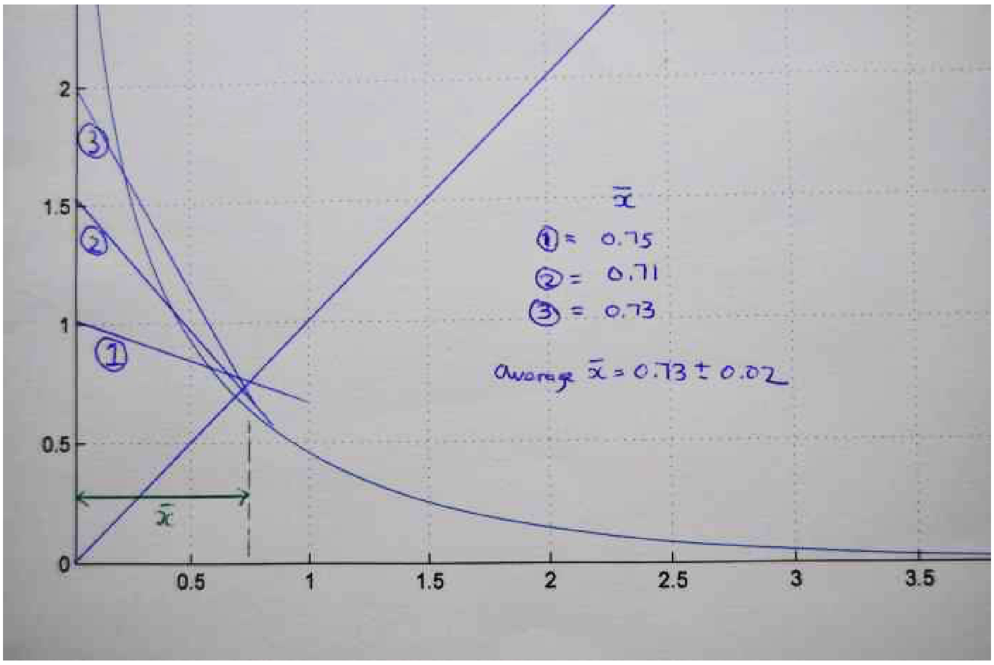

Figure 7: Three independent centroid estimates giving $\bar{x} \approx 0.73 \pm 0.02$.

The principal sources of uncertainty are truncation of the unbounded region, manual cutting of the boundary and reading of the plumb-line intersections. Accordingly, the quoted uncertainty should be regarded as exploratory rather than statistical.

Despite its simplicity, the paper model provides a remarkably good physical illustration of the centroid interpretation of Apéry's constant.

References

[1] M. Passare, *How to Compute* $\sum_{n=1}^{\infty} 1/n^2$ *by Solving Triangles*, The American Mathematical Monthly **115** (2008), no. 8, 745–752.

[2] B. Scannell, *Observations on Zeta(3) from Piling Cubes*, viXra:1603.0414, March 2016.

[3] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, Amsterdam, 1981.