Geometric Interpretations of $\zeta(3)$ from Piling Cubes

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Abstract

We investigate geometric interpretations of $\zeta(3)$ arising from piled cubes and logarithmic surfaces. Motivated by Passare's geometric interpretation of the Basel problem, a surface

$$z = -\log\bigl(1 - e^{-(x+y)}\bigr), \qquad x, y \geq 0,$$

is used to represent $\zeta(3)$ as a volume. The volume identity is proved rigorously using the logarithmic series and Tonelli's theorem. Radial sections, planar cuts, centroid relations and surfaces of revolution are then examined. These constructions give geometric interpretations of standard integral representations for $\zeta(3)$ and related polylogarithmic identities, including formulae for $\mathrm{Li}_2(1/2)$ and $\mathrm{Li}_3(1/2)$. The emphasis is interpretive and geometric: the aim is to give a coherent visual framework for known zeta and polylogarithmic structures.

1 Introduction

Passare's geometric treatment of the Basel problem suggests that values of the Riemann zeta function may sometimes be represented by simple limiting geometric constructions. The Basel value is

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

In Passare's construction, the curve

$$z = -\log(1 - e^{-x})$$

is obtained as the limiting boundary of spread-out squares, and the area beneath this curve is $\zeta(2)$.

The present paper asks for an analogous three-dimensional construction for

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

The motivating picture is that of piled cubes of side lengths

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \ldots,$$

whose total volume is $\zeta(3)$. The central observation is that the continuous limiting object associated with this volume is the logarithmic surface

$$z = -\log\bigl(1 - e^{-(x+y)}\bigr).$$

The integral identities themselves are classical or immediate consequences of standard logarithmic and polylogarithmic expansions. The contribution here is to organise them into a single geometric picture involving piled cubes, radial sections, slices, centroids and volumes of revolution.

2 Passare's curve and the Basel problem

Passare's construction begins with the relation

$$e^{-x} + e^{-z} = 1.$$

Solving for z gives

$$z = -\log(1 - e^{-x}).$$

The resulting curve is shown in Figure 1.

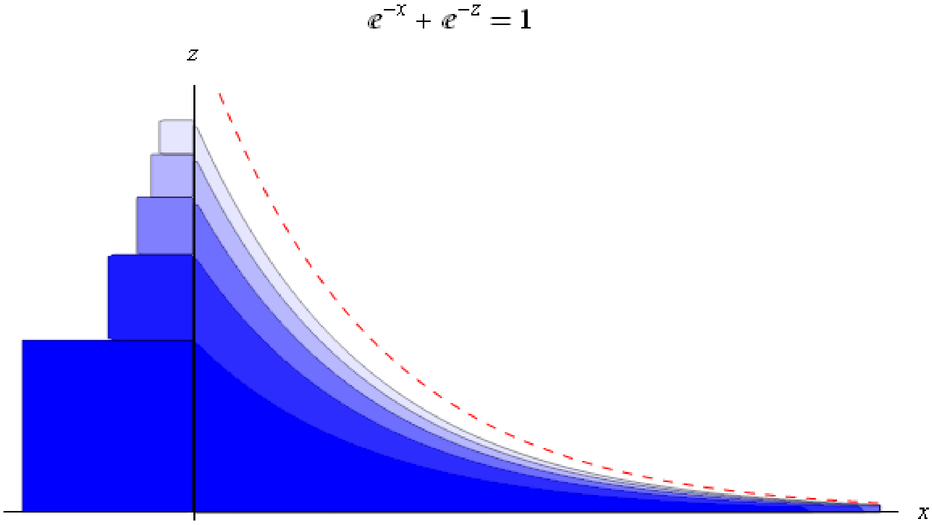

Figure 1: Passare's logarithmic curve obtained as a limiting boundary for spread-out squares.

Proposition 1. *The area under Passare's curve is $\zeta(2)$:*

$$\int_0^\infty -\log(1 - e^{-x})\,dx = \zeta(2).$$

Proof. For $x > 0$,

$$-\log(1 - e^{-x}) = \sum_{n=1}^\infty \frac{e^{-nx}}{n}.$$

All terms are non-negative, so Tonelli's theorem permits termwise integration. Hence

$$\begin{aligned}
\int_0^\infty -\log(1 - e^{-x})\,dx &= \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty e^{-nx}\,dx \\
&= \sum_{n=1}^\infty \frac{1}{n} \cdot \frac{1}{n} \\
&= \sum_{n=1}^\infty \frac{1}{n^2} = \zeta(2).
\end{aligned}$$

□

This gives the two-dimensional starting point for the rest of the paper.

3 Piled cubes and the logarithmic surface for $\zeta(3)$

The extension from squares to cubes is illustrated in Figure 2. The discrete volume

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is represented by a continuous limiting surface.

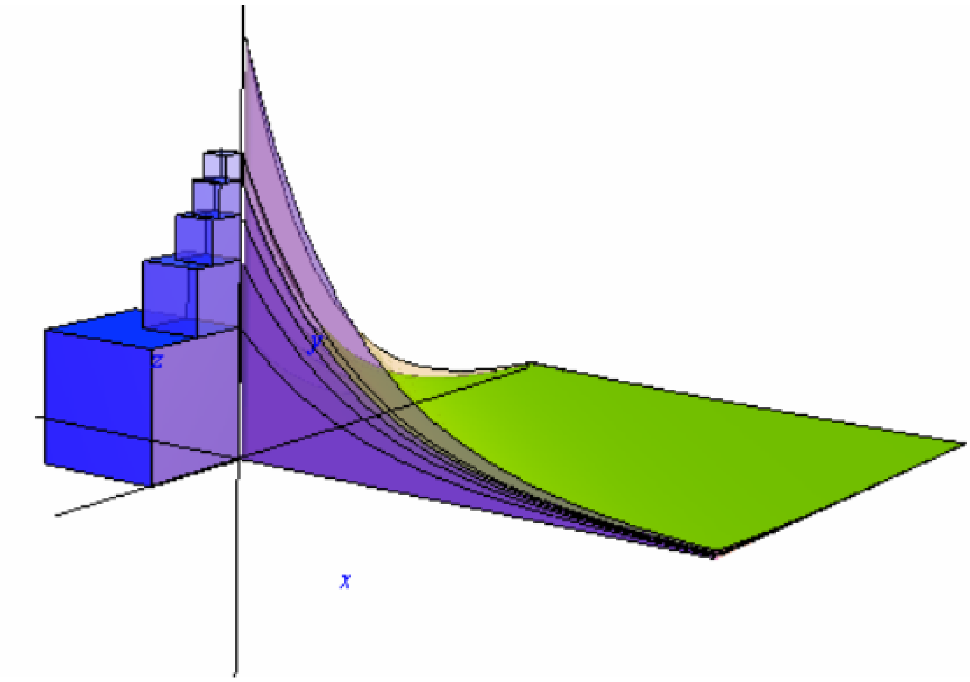

Figure 2: Piled cubes spread out towards a logarithmic surface. Similar shades indicate corresponding volumes.

A first tempting construction would be a surface of revolution of the $\zeta(2)$ curve. However, that construction gives a volume proportional to $\pi\zeta(3)/2$ rather than $\zeta(3)$. This is illustrated in Figure 3.

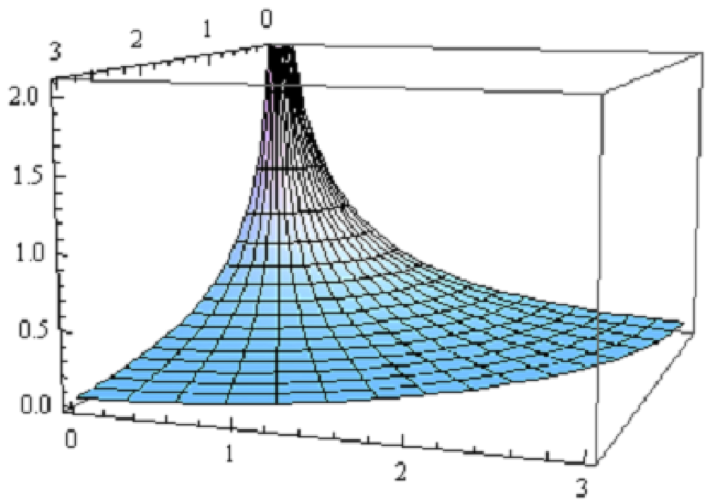

Figure 3: The surface of revolution of the $\zeta(2)$ area gives a volume larger than $\zeta(3)$.

The correct logarithmic surface for the cube volume is instead

$$z = -\log\bigl(1 - e^{-(x+y)}\bigr).$$

Proposition 2. *The surface*

$$z = -\log\bigl(1 - e^{-(x+y)}\bigr), \qquad x, y \geq 0,$$

encloses volume $\zeta(3)$*:*

$$\int_0^\infty \int_0^\infty -\log\bigl(1 - e^{-(x+y)}\bigr)\, dx\, dy = \zeta(3).$$

Proof. For $x, y > 0$,

$$-\log\bigl(1 - e^{-(x+y)}\bigr) = \sum_{n=1}^{\infty} \frac{e^{-n(x+y)}}{n}.$$

Since the summands are non-negative, Tonelli's theorem justifies interchanging the sum and the two integrals. Thus

$$\begin{aligned}
&\int_0^\infty \int_0^\infty -\log\bigl(1 - e^{-(x+y)}\bigr)\, dx\, dy \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n} \int_0^\infty \int_0^\infty e^{-n(x+y)}\, dx\, dy \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^\infty e^{-nx}\, dx\right) \left(\int_0^\infty e^{-ny}\, dy\right) \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n}\right) \left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3).
\end{aligned}$$

□

4 A higher-dimensional form

The same construction extends naturally to higher dimensions.

Proposition 3. *For each integer $k \geq 1$,*

$$\int_{[0,\infty)^k} -\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr)\, dx_1 \cdots dx_k = \zeta(k+1).$$

Proof. For $x_1, \ldots, x_k > 0$,

$$-\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr) = \sum_{n=1}^{\infty} \frac{e^{-n(x_1+\cdots+x_k)}}{n}.$$

Again Tonelli's theorem applies, because the summands are non-negative. Hence

$$\begin{aligned}
&\int_{[0,\infty)^k} -\log\bigl(1 - e^{-(x_1+\cdots+x_k)}\bigr)\, dx_1 \cdots dx_k \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n} \prod_{j=1}^{k} \int_0^{\infty} e^{-nx_j}\, dx_j \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{n}\right)^k \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n^{k+1}} = \zeta(k+1).
\end{aligned}$$

□

Thus the double-integral representation for $\zeta(3)$ is the first nontrivial member of a family of logarithmic volume representations.

5 Radial geometry of the $\zeta(3)$ surface

Introduce polar coordinates in the first quadrant:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

The surface becomes

$$z = -\log\bigl(1 - e^{-r(\cos\theta+\sin\theta)}\bigr).$$

Therefore

$$\zeta(3) = \int_0^{\pi/2} \int_0^{\infty} -\log\bigl(1 - e^{-r(\cos\theta+\sin\theta)}\bigr) r\, dr\, d\theta.$$

For a fixed angle θ, the area under the radial section is

$$A(\theta) = \int_0^{\infty} -\log\bigl(1 - e^{-r(\cos\theta+\sin\theta)}\bigr)\, dr.$$

By the substitution $u = r(\cos\theta + \sin\theta)$,

$$A(\theta) = \frac{\zeta(2)}{\cos\theta + \sin\theta}.$$

Hence the radial section is largest at the coordinate axes and smallest on the diagonal $\theta = \pi/4$. Figure 4 shows this normalised radial area.

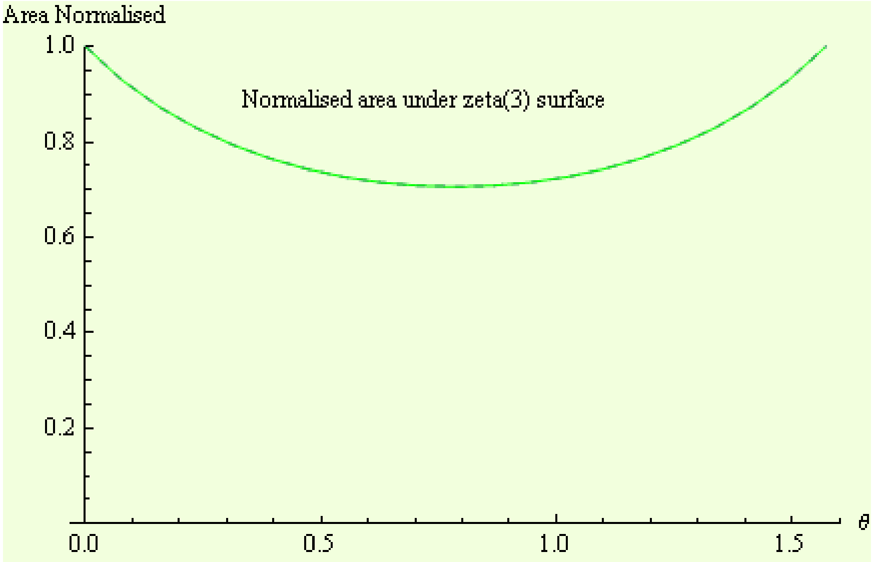

Figure 4: The radial area under the $\zeta(3)$ surface, normalised by $\zeta(2)$, is $1/(\cos\theta + \sin\theta)$.

The corresponding three-dimensional views of the surface are shown in Figure 5.

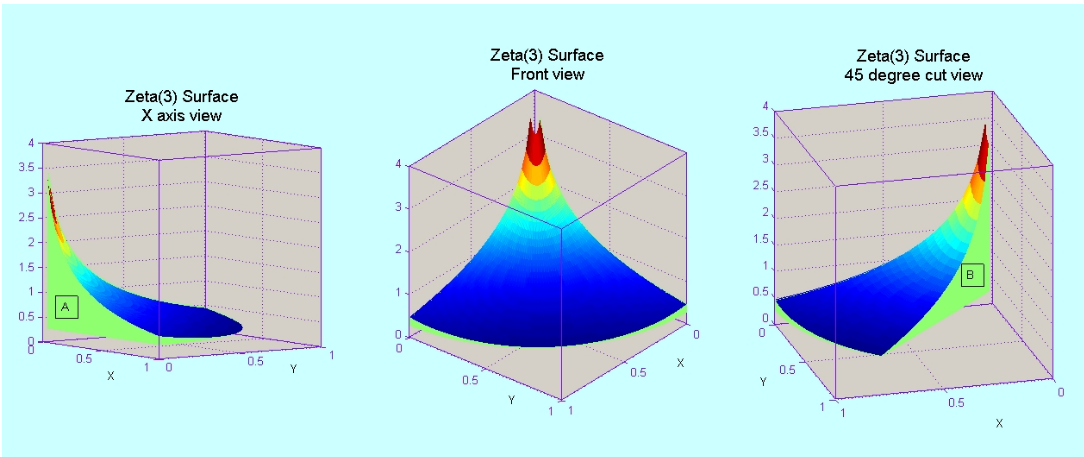

Figure 5: Views of the $\zeta(3)$ surface. The radial sections dip towards the diagonal section $\theta = \pi/4$.

6 Radial sections and cube sections

The surface has the correct total volume $\zeta(3)$, but the areas of its radial sections do not directly match the rectangular or square sections of the stacked cubes. This distinction is important. Figure 6 shows the plan view of the surface with a diagonal section of a cube.

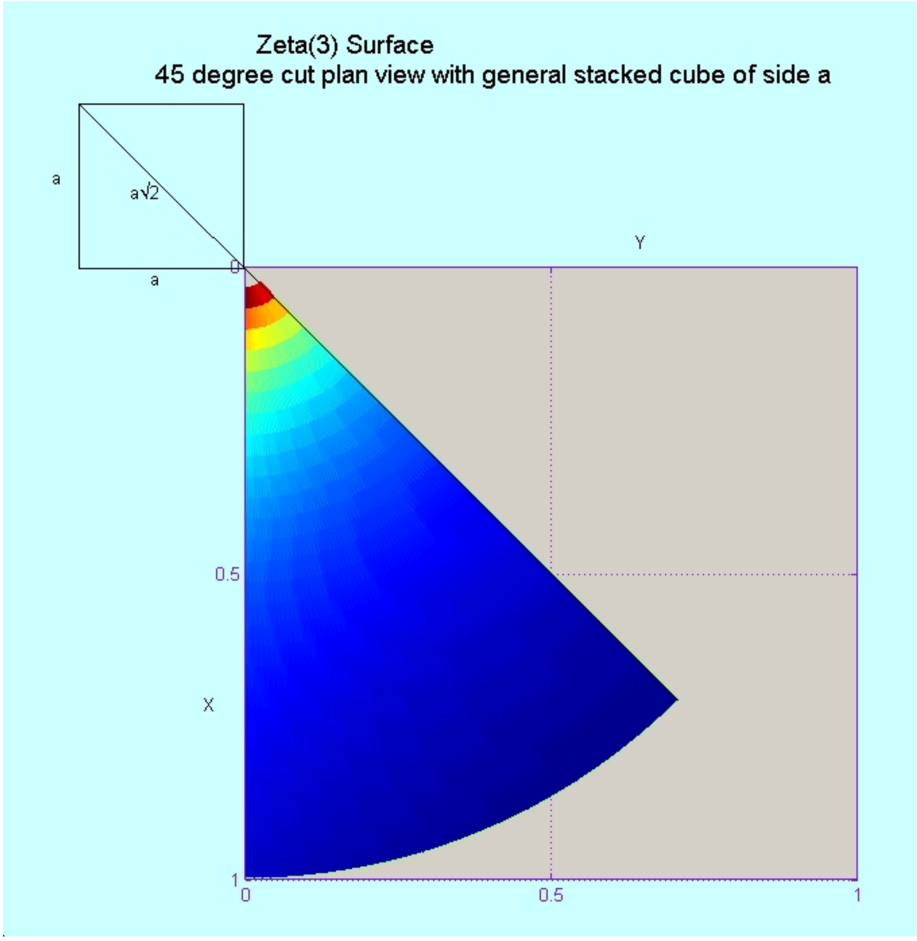

Figure 6: Plan view of the $\zeta(3)$ surface together with a diagonal section of a cube.

Along the diagonal $x = y$, the corresponding cube sections scale by a factor involving $\sqrt{2}$, whereas the radial section of the logarithmic surface scales inversely. This explains why exact volume correspondence and exact section-by-section correspondence are different geometric requirements.

Figure 7 shows the $x = y$ section together with the corresponding stacked cube sections.

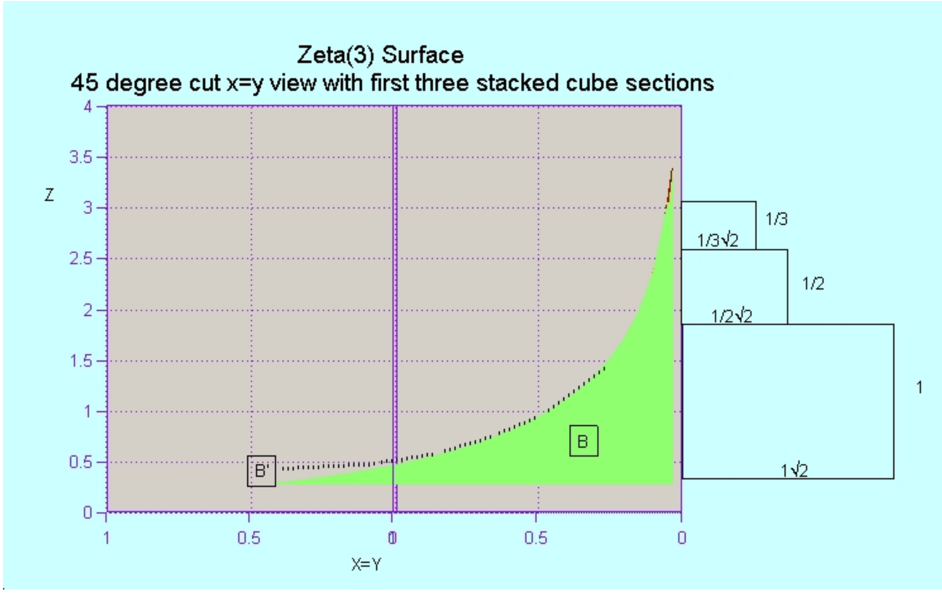

Figure 7: The $x = y$ section of the $\zeta(3)$ surface together with stacked cube sections.

Thus the logarithmic surface correctly represents the total volume $\zeta(3)$, but does not reproduce every planar cube section exactly.

7 Alternative logarithmic surfaces

It is useful to compare three related surfaces:

1. the volume-preserving surface

$$z = -\log\big(1 - e^{-r(\cos\theta+\sin\theta)}\big),$$

   whose volume is $\zeta(3)$;

2. the surface of revolution

$$z = -\log(1 - e^{-r}),$$

   whose quadrant volume is $\frac{\pi}{2}\zeta(3)$;

3. the section-preserving surface

$$z = -\log\big(1 - e^{-r/(\cos\theta+\sin\theta)}\big),$$

   which reproduces the scaling of the radial cube sections but gives a larger total volume.

For the third surface,

$$\int_0^\infty -\log(1 - e^{-ar})\, r\, dr = \frac{\zeta(3)}{a^2}.$$

Taking

$$a = \frac{1}{\cos\theta + \sin\theta}$$

gives

$$\int_0^\infty -\log\left(1-e^{-r/(\cos\theta+\sin\theta)}\right) r\,dr = \zeta(3)(\cos\theta+\sin\theta)^2.$$

Therefore its total volume is

$$\begin{aligned}\zeta(3)\int_0^{\pi/2}(\cos\theta+\sin\theta)^2\,d\theta &= \zeta(3)\int_0^{\pi/2}(1+\sin 2\theta)\,d\theta \\ &= \left(\frac{\pi}{2}+1\right)\zeta(3).\end{aligned}$$

This is larger than $\zeta(3)$ by the factor $\pi/2+1$.

The comparison is shown visually in Figures 8–10.

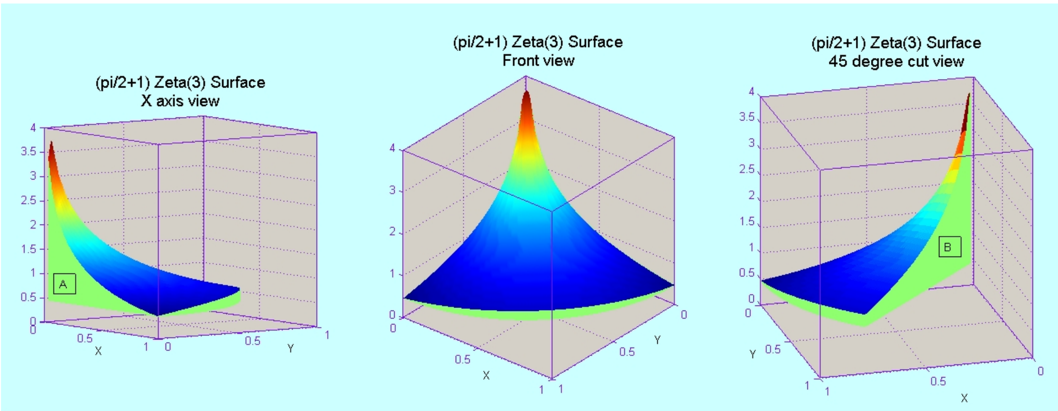

Figure 8: A section-preserving logarithmic surface. This surface peaks at $\theta = \pi/4$ rather than dipping there.

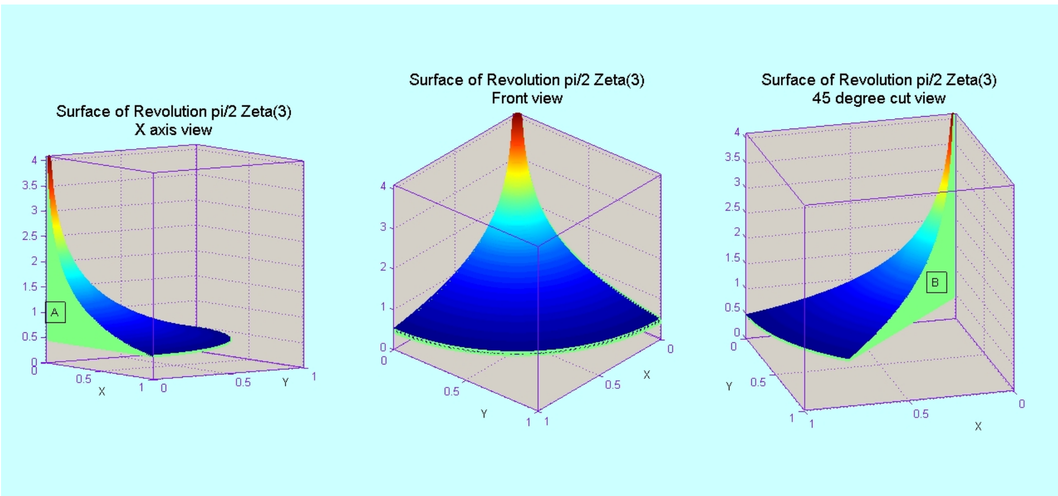

Figure 9: The surface of revolution of the $\zeta(2)$ curve.

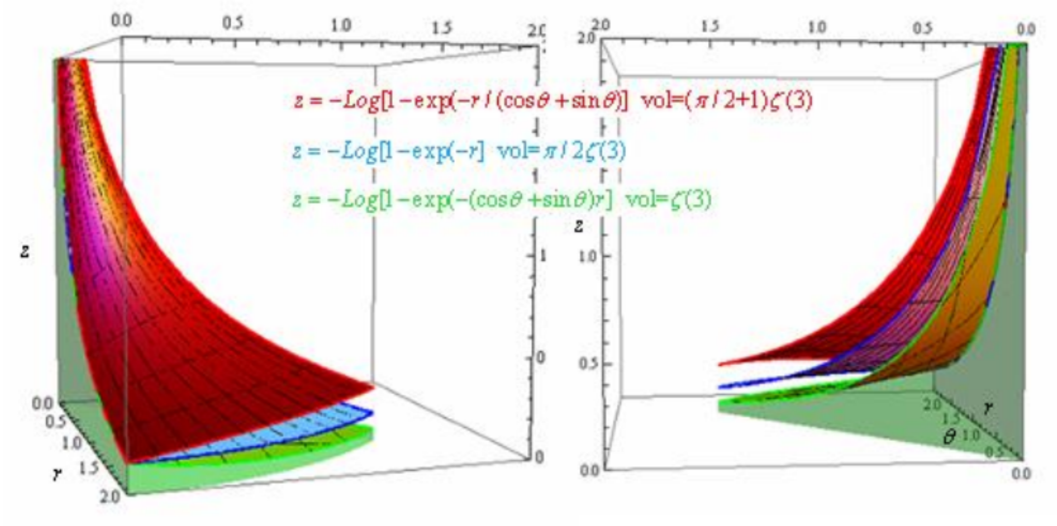

Figure 10: Comparison of logarithmic surfaces giving different multiples of $\zeta(3)$.

Subtracting one volume from another may itself produce an alternative region of volume $\zeta(3)$, as illustrated in Figure 11.

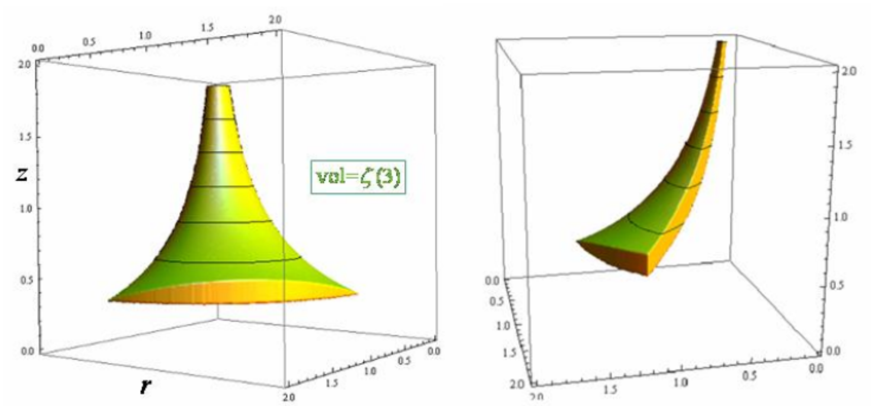

Figure 11: Part of an alternative shape whose volume is $\zeta(3)$.

8 Centroid and moment interpretations

Although the main construction is three-dimensional, $\zeta(3)$ also appears naturally as a first moment of the original two-dimensional $\zeta(2)$ region.

Let A be the region under

$$y = -\log(1 - e^{-x}), \qquad x \geq 0.$$

Its area is $\zeta(2)$. The x-coordinate of its centroid is

$$\bar{x} = \frac{\displaystyle\int_0^\infty x[-\log(1 - e^{-x})]\, dx}{\displaystyle\int_0^\infty [-\log(1 - e^{-x})]\, dx}.$$

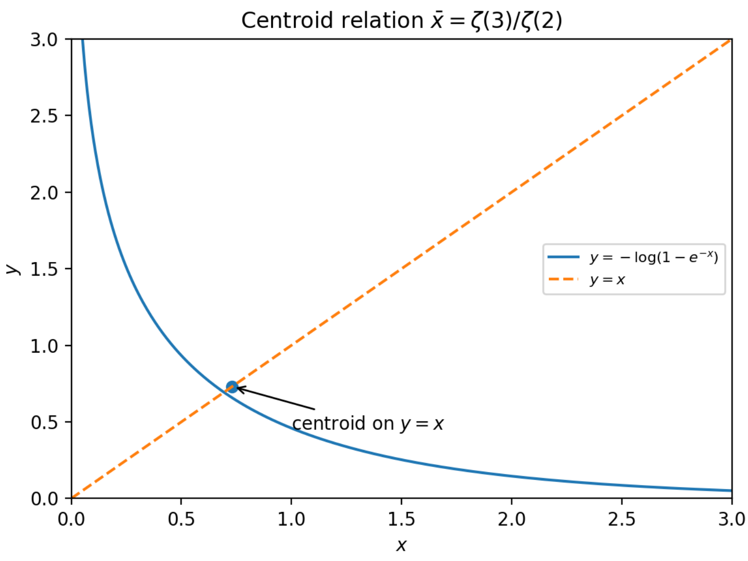

Figure 12: The centroid relation for the plane region under $y = -\log(1 - e^{-x})$.

Proposition 4.

$$\bar{x} = \frac{\zeta(3)}{\zeta(2)}.$$

Equivalently,

$$\zeta(3) = \bar{x}\zeta(2).$$

Proof. We already have

$$\int_0^\infty -\log(1 - e^{-x})\,dx = \zeta(2).$$

For the numerator,

$$\int_0^\infty x[-\log(1 - e^{-x})]\,dx = \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty xe^{-nx}\,dx.$$

Since

$$\int_0^\infty xe^{-nx}\,dx = \frac{1}{n^2},$$

we get

$$\int_0^\infty x[-\log(1 - e^{-x})]\,dx = \sum_{n=1}^\infty \frac{1}{n^3} = \zeta(3).$$

The claimed centroid formula follows. □

The second moment of the same region also gives a related identity:

$$\int_0^\infty x^2[-\log(1 - e^{-x})]\,dx = 2\zeta(4).$$

This follows from

$$\int_0^\infty x^2 e^{-nx}\,dx = \frac{2}{n^3}.$$

Thus $\zeta(4)$ appears as a moment of the same logarithmic region.

9 Polylogarithmic slices

The $\zeta(3)$ surface naturally generates polylogarithms when it is cut by planes parallel to the coordinate axes. This is illustrated in Figure 13.

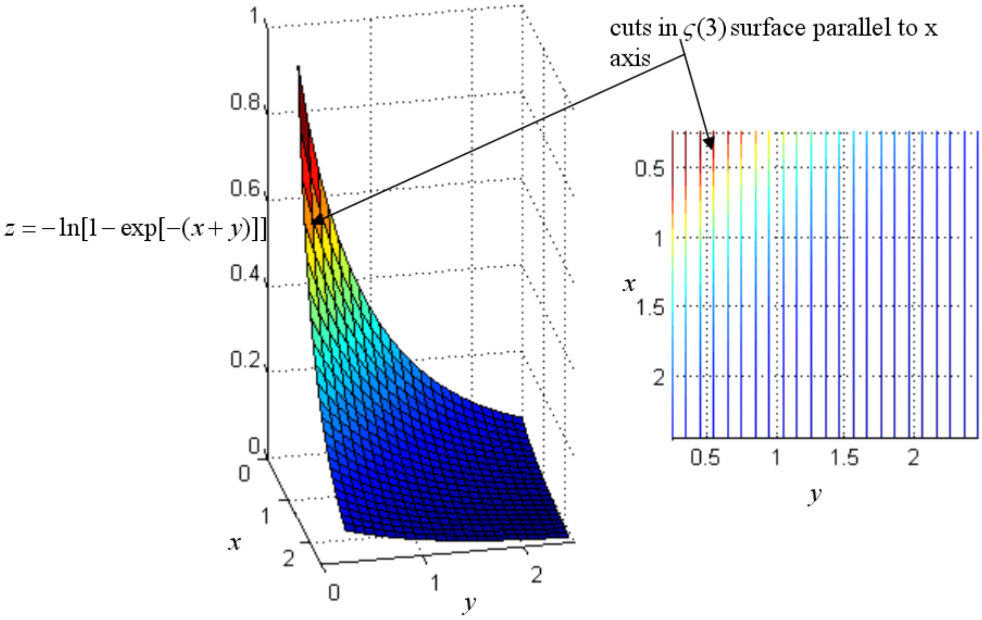

Figure 13: Areas under slices of the $\zeta(3)$ surface parallel to the x-axis generate dilogarithms.

For fixed $y \geq 0$,

$$\begin{aligned}\int_0^\infty -\log(1-e^{-(x+y)})\,dx &= \sum_{n=1}^\infty \frac{e^{-ny}}{n}\int_0^\infty e^{-nx}\,dx \\ &= \sum_{n=1}^\infty \frac{e^{-ny}}{n^2} \\ &= \mathrm{Li}_2(e^{-y}).\end{aligned}$$

Integrating these slice areas over y recovers the volume:

$$\zeta(3) = \int_0^\infty \mathrm{Li}_2(e^{-y})\,dy.$$

More generally, for $m > 1$,

$$\int_0^\infty \mathrm{Li}_{m-1}(e^{-y})\,dy = \zeta(m).$$

Indeed,

$$\begin{aligned}\int_0^\infty \mathrm{Li}_{m-1}(e^{-y})\,dy &= \int_0^\infty \sum_{n=1}^\infty \frac{e^{-ny}}{n^{m-1}}\,dy \\ &= \sum_{n=1}^\infty \frac{1}{n^{m-1}}\int_0^\infty e^{-ny}\,dy \\ &= \sum_{n=1}^\infty \frac{1}{n^m} = \zeta(m).\end{aligned}$$

10 Sections at constant height

Cuts of the $\zeta(3)$ surface at constant height z form triangular regions in the xy-plane. The geometry is shown in Figure 14.

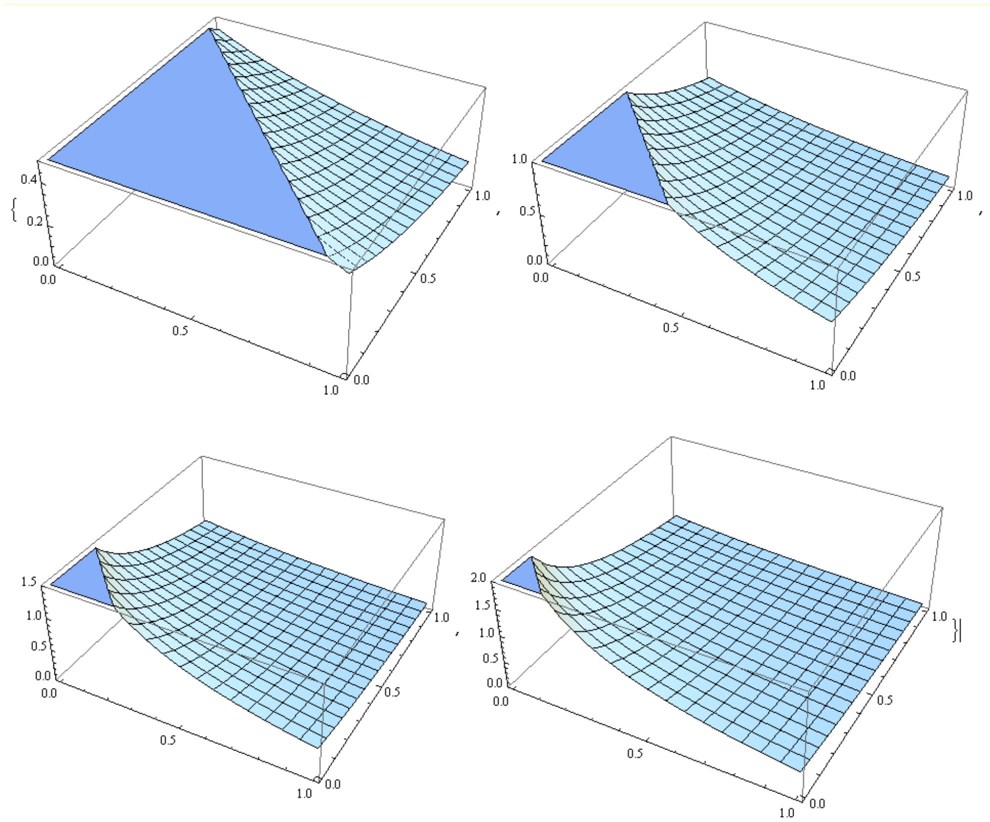

Figure 14: Constant-z sections of the $\zeta(3)$ surface form triangular regions.

The boundary condition

$$z = -\log(1 - e^{-(x+y)})$$

is equivalent to

$$x + y = -\log(1 - e^{-z}).$$

Thus the constant-z cross-section is a right triangle of side length

$$-\log(1 - e^{-z}).$$

Its area is therefore

$$\frac{1}{2}[-\log(1 - e^{-z})]^2.$$

Integrating the cross-sectional area over z gives

$$\zeta(3) = \frac{1}{2}\int_0^\infty [\log(1 - e^{-z})]^2\, dz.$$

This is the logarithmic version of a standard integral representation for $\zeta(3)$.

11 Geometric derivation of $\mathrm{Li}_2(1/2)$

The curve

$$y = -\log(1 - e^{-x})$$

is symmetric about the line $y = x$. The intersection with this line satisfies

$$x = -\log(1 - e^{-x}).$$

Equivalently,

$$e^{-x} = 1 - e^{-x},$$

so

$$e^{-x} = \frac{1}{2}, \qquad x = \log 2.$$

The geometry is shown in Figure 15. The square of side length $\log 2$ gives the central symmetric contribution, while the remaining part corresponds to $\mathrm{Li}_2(1/2)$.

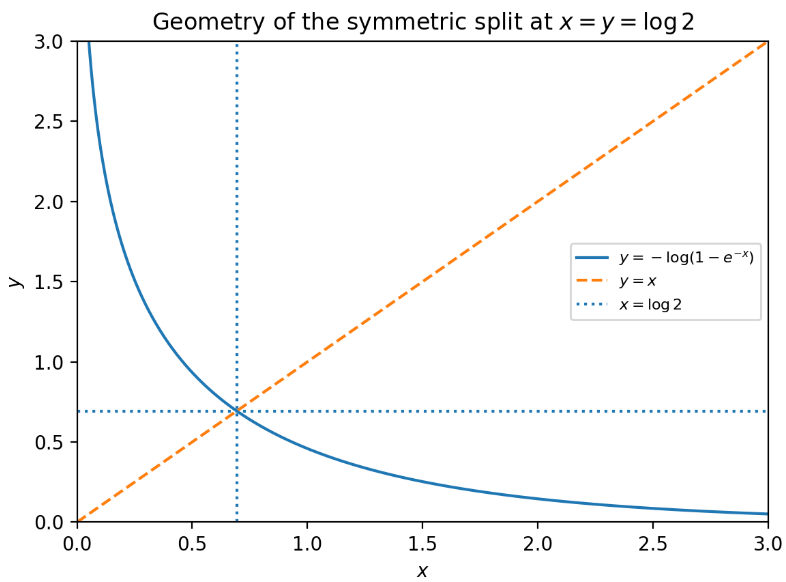

Figure 15: Geometry of the symmetric split at $x = y = \log 2$, leading to the classical formula for $\mathrm{Li}_2(1/2)$.

The total area under the curve is $\zeta(2) = \pi^2/6$. By symmetry,

$$\zeta(2) = 2\,\mathrm{Li}_2(1/2) + (\log 2)^2.$$

Therefore

$$\mathrm{Li}_2(1/2) = \frac{1}{2}\zeta(2) - \frac{1}{2}(\log 2)^2 = \frac{\pi^2}{12} - \frac{(\log 2)^2}{2}.$$

12 Geometric derivation of $\mathrm{Li}_3(1/2)$

The corresponding volume construction for $\mathrm{Li}_3(1/2)$ is obtained by rotating the same plane region through a quadrant. The geometric splitting is shown in Figures 16–19.

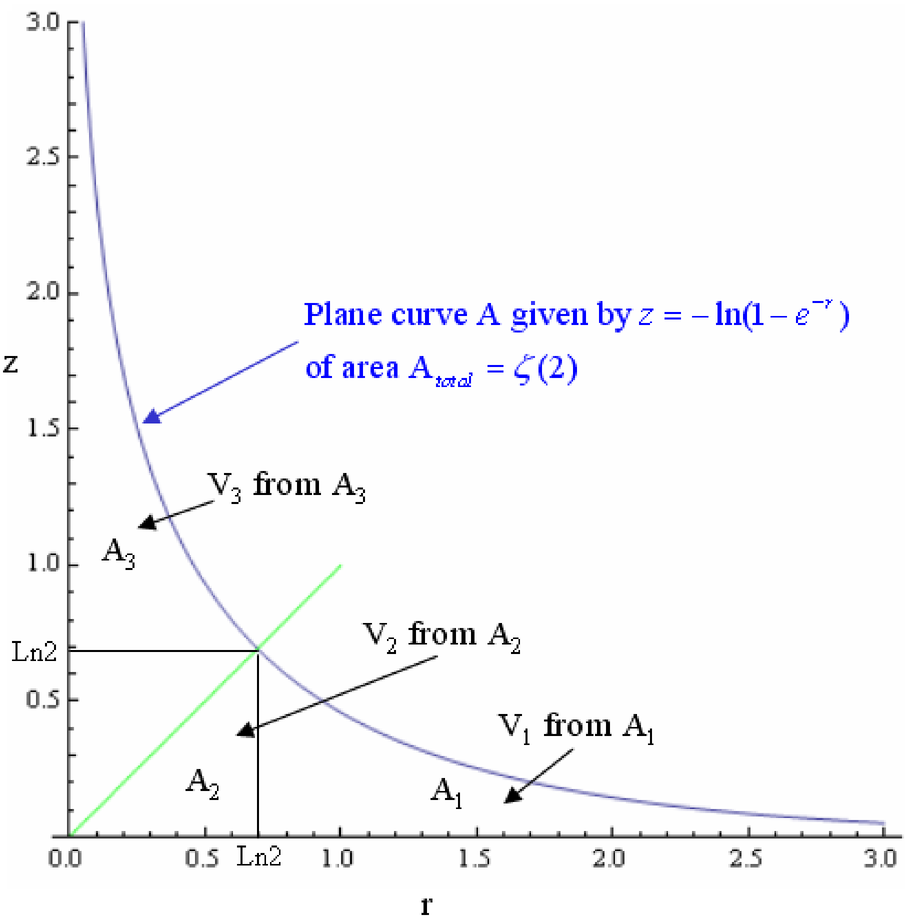

Figure 16: Splitting the $\zeta(2)$ region into subregions associated with volumes of revolution.

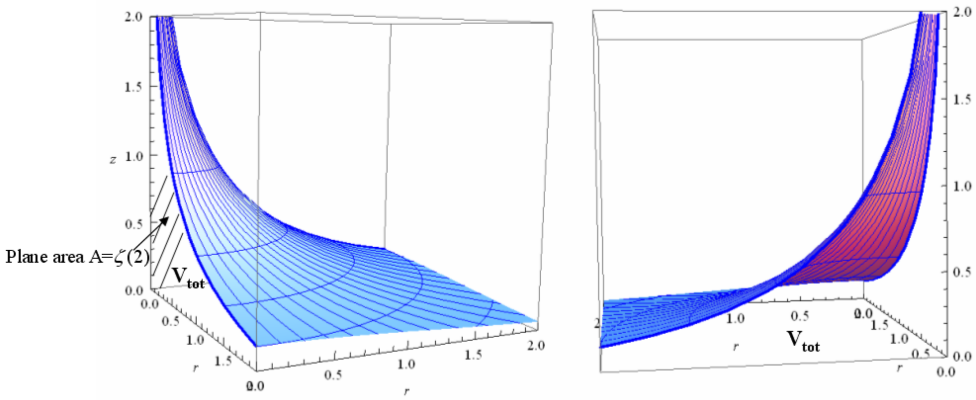

Figure 17: The quadrant volume of revolution associated with the plane $\zeta(2)$ region.

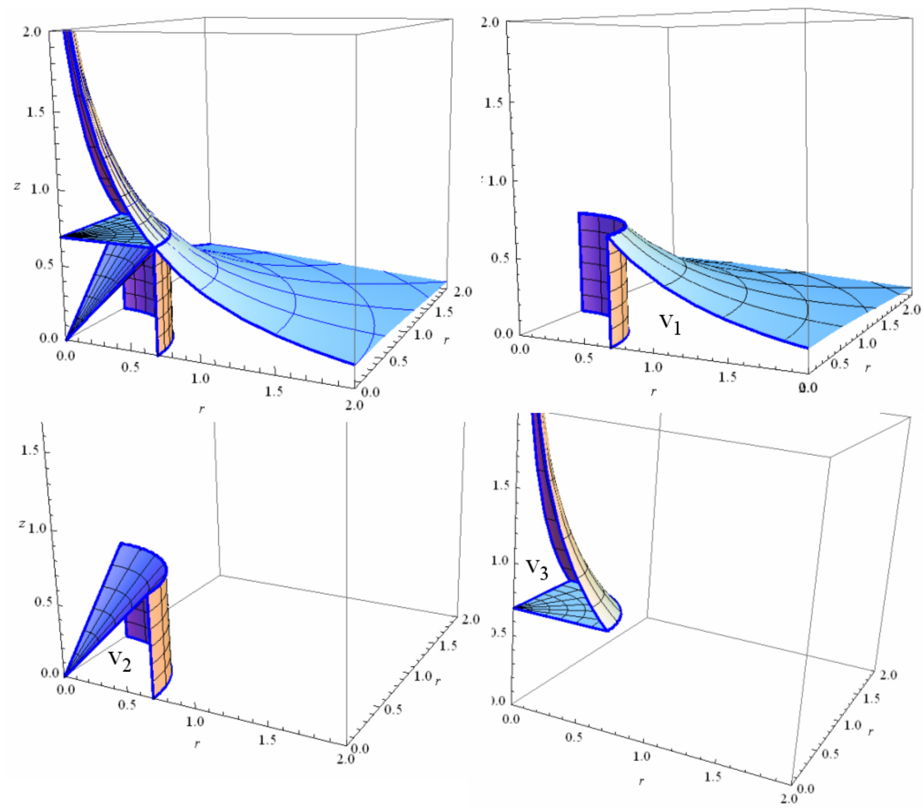

Figure 18: Volumes of revolution corresponding to the subregions in the split $\zeta(2)$ area.

For $a \geq 0$, define

$$I(a) = \int_a^\infty r[-\log(1-e^{-r})]\,dr.$$

Expanding the logarithm gives

$$\begin{aligned} I(a) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_a^\infty r e^{-nr}\,dr \\ &= \sum_{n=1}^{\infty} \frac{1}{n} e^{-na} \left(\frac{a}{n} + \frac{1}{n^2} \right) \\ &= a\,\mathrm{Li}_2(e^{-a}) + \mathrm{Li}_3(e^{-a}). \end{aligned}$$

Taking $a = \log 2$ gives

$$I(\log 2) = (\log 2)\,\mathrm{Li}_2(1/2) + \mathrm{Li}_3(1/2).$$

Combining the corresponding volume pieces gives the classical identity

$$\mathrm{Li}_3(1/2) = \frac{7}{8}\zeta(3) - \frac{\pi^2 \log 2}{12} + \frac{(\log 2)^3}{6}.$$

The decomposition is summarised geometrically in Figure 19.

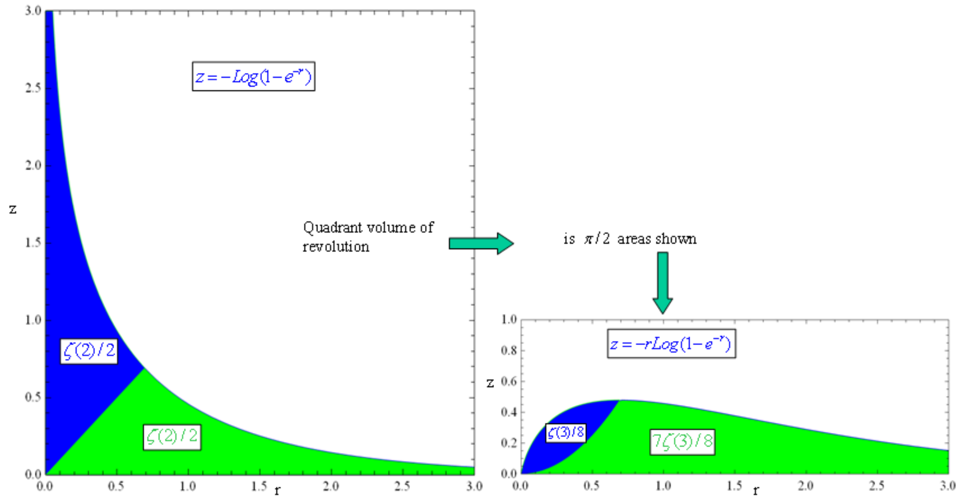

Figure 19: The symmetric plane areas are transformed into corresponding $\zeta(3)$ volumes under a quadrant of revolution.

13 Conclusion

The piled-cube construction leads naturally to the surface

$$z = -\log\bigl(1 - e^{-(x+y)}\bigr),$$

whose enclosed volume is $\zeta(3)$. This surface provides a common geometric setting for several related structures:

- logarithmic integral representations of $\zeta(2)$ and $\zeta(3)$;
- higher-dimensional integral representations of $\zeta(k+1)$;
- radial sections whose areas vary as $\zeta(2)/(\cos\theta + \sin\theta)$;
- polylogarithmic slices involving $\mathrm{Li}_2(e^{-y})$;
- triangular height sections giving a squared-logarithm representation of $\zeta(3)$;
- centroid and moment interpretations of zeta values;
- geometric decompositions leading to the classical values of $\mathrm{Li}_2(1/2)$ and $\mathrm{Li}_3(1/2)$.

The main purpose of the paper is not to claim fundamentally new zeta identities, but to organise known analytic identities into a coherent geometric and visual framework suggested by piled squares and piled cubes.

A Experimental determination of the centroid

The following experiment is included as an illustrative physical verification of the centroid identity

$$\bar{x} = \frac{\zeta(3)}{\zeta(2)}.$$

It is not used as mathematical evidence for the identity, which was proved rigorously above.

The region under

$$y = -\log(1 - e^{-x})$$

was plotted, printed with equal scaling on the axes, and cut from thick paper. The centroid was estimated by suspending the cut-out and drawing plumb lines. The experimental method is shown in Figures 20–22.

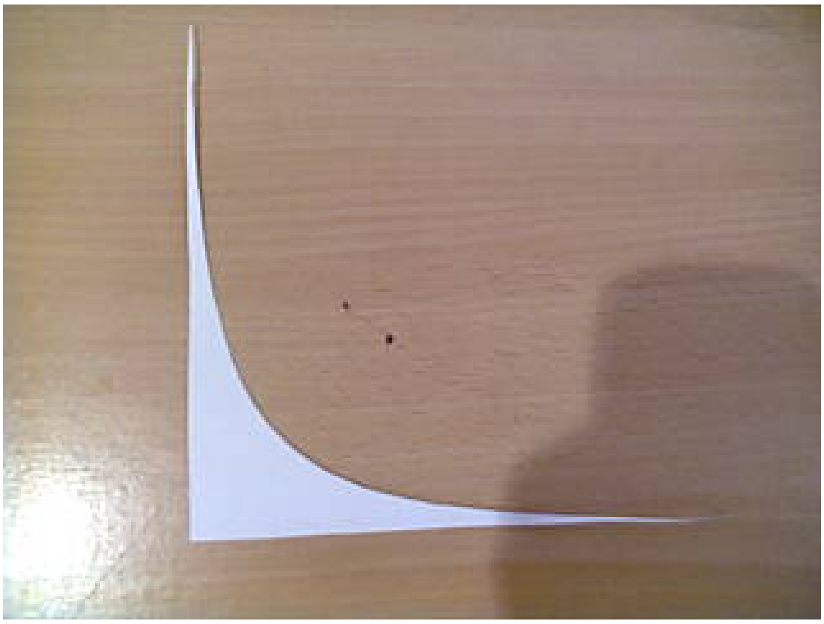

Figure 20: Cut-out of the plane region under $y = -\log(1 - e^{-x})$.

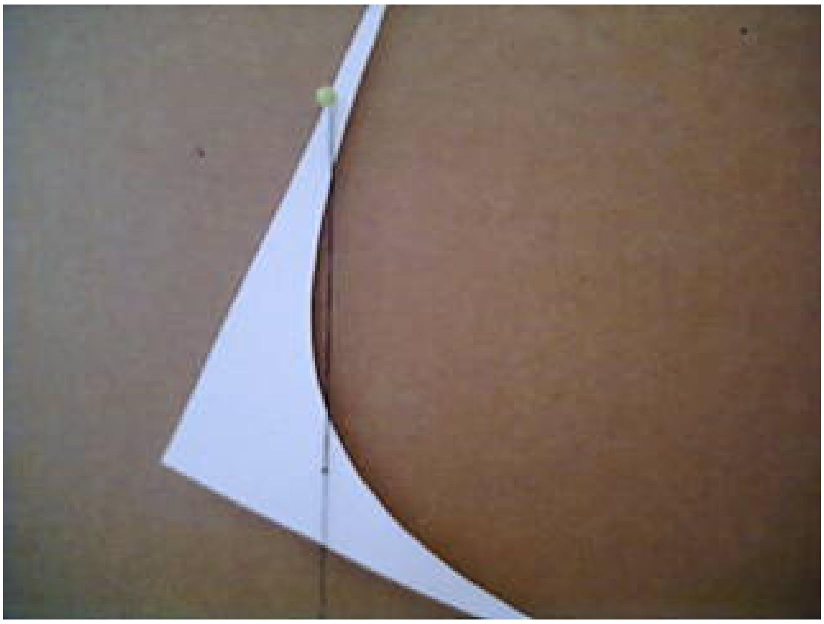

Figure 21: Estimating the centroid using a plumb line.

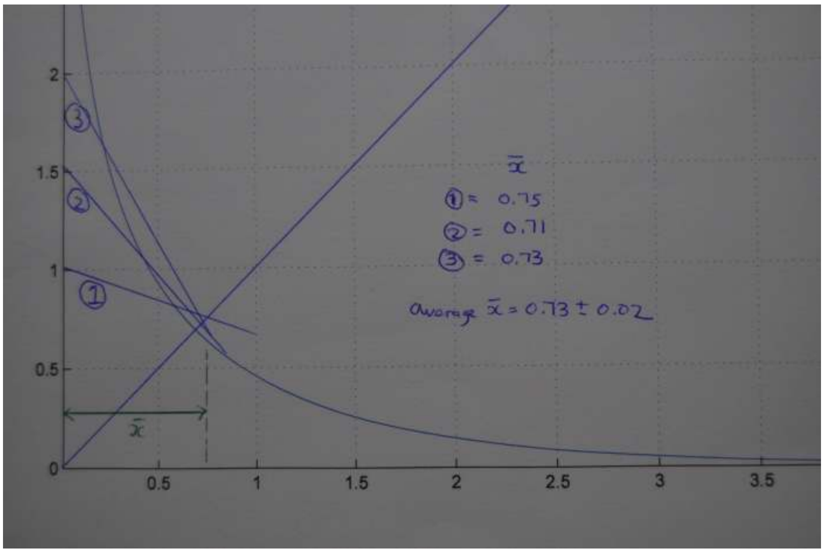

Figure 22: Measured centroid lines compared with the expected centroid on $y = x$.

The original measurements gave an estimate near

$$\bar{x} \approx 0.73,$$

and hence

$$\zeta(3) \approx \bar{x}\zeta(2) \approx 1.20,$$

consistent with the known value

$$\zeta(3) = 1.2020569\ldots.$$

B Standard logarithmic and polylogarithmic identities

The polylogarithm is defined by

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

For $z = 1$ this gives

$$\mathrm{Li}_s(1) = \zeta(s).$$

The basic logarithmic identity used throughout is

$$-\log(1-u) = \sum_{n=1}^{\infty} \frac{u^n}{n}, \qquad |u| < 1.$$

It implies, for $a > 0$ and $m \geq 0$,

$$\int_0^\infty x^m[-\log(1-e^{-ax})]\,dx = \frac{\Gamma(m+1)}{a^{m+1}}\zeta(m+2).$$

This follows by expanding the logarithm and using

$$\int_0^\infty x^m e^{-anx}\,dx = \frac{\Gamma(m+1)}{(an)^{m+1}}.$$

In particular,

$$\int_0^\infty -\log(1 - e^{-ax})\,dx = \frac{\zeta(2)}{a},$$

and

$$\int_0^\infty x[-\log(1 - e^{-ax})]\,dx = \frac{\zeta(3)}{a^2}.$$

References

[1] M. Passare, *How to compute $\sum 1/n^2$ by plotting triangles*, arXiv:math/0701039.

[2] B. Riemann, *On the Number of Prime Numbers Less Than a Given Magnitude*, 1859.