Many Angles, One Integer Sequence

Brian Scannell

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Abstract

The irrational numbers

$$\tan 15^\circ = 2 - \sqrt{3}, \qquad \cot 15^\circ = 2 + \sqrt{3}$$

generate the integer sequence

$$2,\ 4,\ 14,\ 52,\ 194, \ldots$$

when corresponding powers are added. Surprisingly, 15° is only one of infinitely many angles that generate this same sequence when the exponents are chosen appropriately.We describe a two-parameter family behind this phenomenon and show how two elementary recurrence relations connect symmetric trigonometric powers with Chebyshev polynomials, Lucas sequences, Pell equations, and, unexpectedly, the Lucas–Lehmer test for Mersenne primes.

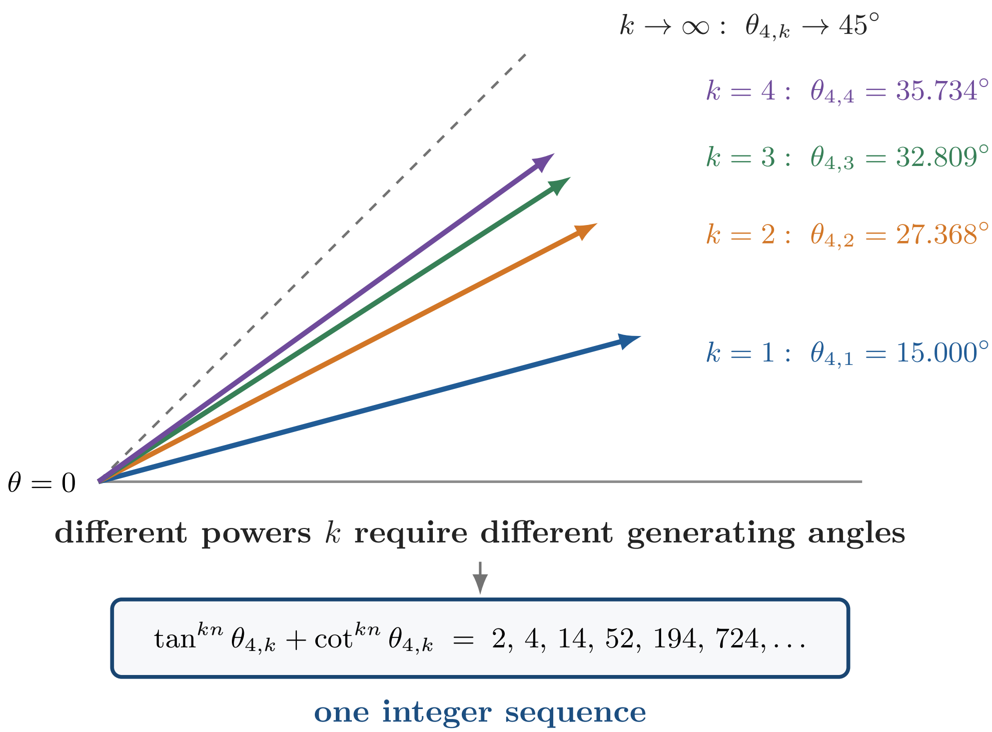

Figure 1: For fixed $M = 4$, different powers k require different generating angles $\theta_{4,k}$, yet every choice produces the same sequence $2, 4, 14, 52, 194, \ldots$ when the exponent is kn.

1 An integer sequence from 15°

Consider

$$\tan 15^\circ = 2-\sqrt{3}, \qquad \cot 15^\circ = 2+\sqrt{3}.$$

Both numbers are irrational, but their sum is 4. More surprisingly, if

$$A_n = \tan^n 15^\circ + \cot^n 15^\circ,$$

then

$$A_n : \qquad 2,\ 4,\ 14,\ 52,\ 194,\ 724,\ 2702,\ldots$$

and every term is an integer.

For example,

$$(2+\sqrt{3})^2 + (2-\sqrt{3})^2 = 14,$$

and

$$(2+\sqrt{3})^3 + (2-\sqrt{3})^3 = 52.$$

The cancellation of the radicals is explained by algebraic conjugacy, but there is a more useful way to see the persistence of integrality.

Let

$$a = 2+\sqrt{3}.$$

Then

$$a^{-1} = 2-\sqrt{3}$$

and

$$a + a^{-1} = 4.$$

Hence

$$a^2 - 4a + 1 = 0.$$

Multiplying by a^{n-2} gives

$$a^n = 4a^{n-1} - a^{n-2}.$$

The same relation holds for a^{-1}, so

$$A_n = 4A_{n-1} - A_{n-2}, \tag{1}$$

with

$$A_0 = 2, \qquad A_1 = 4.$$

The integer sequence is therefore produced by a second-order recurrence. This suggests a broader question: *which angles generate integer sequences in this way?*

2 Symmetric powers

For $z > 0$, define

$$S_n(z) = z^n + z^{-n},$$

and put

$$s = z + z^{-1}.$$

Since z and z^{-1} are roots of

$$x^2 - sx + 1 = 0,$$

we obtain

$$S_n(z) = sS_{n-1}(z) - S_{n-2}(z), \tag{2}$$

with

$$S_0(z) = 2, \qquad S_1(z) = s.$$

Thus

$$S_2 = s^2 - 2, \qquad S_3 = s^3 - 3s, \qquad S_4 = s^4 - 4s^2 + 2.$$

Equation (2) will appear twice, in two different roles.

3 First recurrence direction: vary the power

Fix an angle

$$0 < \theta < \frac{\pi}{2}$$

and define

$$M_k(\theta) = \tan^k \theta + \cot^k \theta, \qquad k \geq 0.$$

Let

$$x = \tan\theta + \cot\theta.$$

Applying (2) to $z = \tan\theta$ gives

$$M_0 = 2, \qquad M_1 = x,$$

and

$$M_k = xM_{k-1} - M_{k-2}. \tag{3}$$

The first few terms are therefore

$$M_2 = x^2 - 2,$$

$$M_3 = x^3 - 3x,$$

$$M_4 = x^4 - 4x^2 + 2.$$

These are Chebyshev polynomials in disguise. Using the standard identity [2],

$$T_k\left(\frac{z + z^{-1}}{2}\right) = \frac{z^k + z^{-k}}{2}$$

gives

$$\tan^k \theta + \cot^k \theta = 2T_k\left(\frac{\tan\theta + \cot\theta}{2}\right). \tag{4}$$

This explains why 22.5° almost behaves like 15°. Since

$$\tan 22.5^\circ + \cot 22.5^\circ = 2\sqrt{2},$$

the first power is not integral. But

$$M_2 = (2\sqrt{2})^2 - 2 = 6,$$

so

$$\tan^2 22.5^\circ + \cot^2 22.5^\circ = 6.$$

The natural question is therefore not only when $\tan\theta + \cot\theta$ is an integer, but when $\tan^k \theta + \cot^k \theta$ is an integer for some k.

4 Second recurrence direction: fix the power

Suppose that for some positive integer k,

$$\tan^k \theta + \cot^k \theta = M, \tag{5}$$

where M is an integer.

Now define

$$A_n = \tan^{kn} \theta + \cot^{kn} \theta.$$

Set

$$a = \tan^k \theta.$$

Then

$$a^{-1} = \cot^k \theta$$

and

$$a + a^{-1} = M.$$

Thus a and a^{-1} are roots of

$$x^2 - Mx + 1 = 0.$$

Applying (2) again gives

$$A_0 = 2, \qquad A_1 = M,$$

and

$$A_n = MA_{n-1} - A_{n-2}. \tag{6}$$

Hence every A_n is an integer.

The same recurrence structure has therefore appeared in two directions:

$$\theta \longrightarrow M_k(\theta) \longrightarrow A_n.$$

For fixed θ, varying k gives the Chebyshev recurrence (3). Once one value $M_k = M$ is integral, fixing k and varying n gives the integer recurrence (6).

5 Which angles give a prescribed integer?

Let

$$M \geq 2, \qquad k \geq 1$$

be integers. We seek

$$\tan^k \theta + \cot^k \theta = M.$$

Restrict first to

$$0 < \theta \leq \frac{\pi}{4}.$$

Write

$$\tan \theta = e^{-u}, \qquad u \geq 0.$$

Then

$$\cot \theta = e^u,$$

so

$$\tan^k \theta + \cot^k \theta = e^{-ku} + e^{ku} = 2\cosh(ku).$$

Hence

$$2\cosh(ku) = M,$$

which gives

$$u = \frac{1}{k} \operatorname{arcosh} \frac{M}{2}.$$

Therefore

$$\tan \theta_{M,k} = \exp\left[-\frac{1}{k} \operatorname{arcosh}\left(\frac{M}{2}\right)\right], \tag{7}$$

or equivalently

$$\theta_{M,k} = \arctan\left\{\exp\left[-\frac{1}{k} \operatorname{arcosh}\left(\frac{M}{2}\right)\right]\right\}. \tag{8}$$

Since $\cosh(ku)$ is strictly increasing for $u \geq 0$, this solution is unique in $0 < \theta \leq \pi/4$. The complementary angle gives the reciprocal tangent and therefore the same symmetric powers.

Theorem 1. *Let $M \geq 2$ and $k \geq 1$ be integers. Up to replacing θ by $\pi/2 - \theta$, there is a unique angle satisfying*

$$\tan^k \theta + \cot^k \theta = M.$$

It is given by (8).

For this angle,

$$A_n = \tan^{kn} \theta + \cot^{kn} \theta$$

is an integer for every $n \geq 0$, and

$$A_0 = 2, \qquad A_1 = M, \qquad A_n = MA_{n-1} - A_{n-2}.$$

Conversely, if

$$\tan^{kn} \theta + \cot^{kn} \theta$$

is an integer for every $n \geq 1$, then its $n = 1$ term is an integer $M \geq 2$, so θ belongs to this family.

6 Two familiar examples

15°

For $k = 1$,

$$\tan \theta + \cot \theta = \frac{2}{\sin 2\theta}.$$

Thus

$$\tan \theta + \cot \theta = M$$

is equivalent to

$$\sin 2\theta = \frac{2}{M},$$

and hence

$$\theta_{M,1} = \frac{1}{2} \arcsin \frac{2}{M}. \tag{9}$$

Taking $M = 4$,

$$\theta_{4,1} = \frac{1}{2} \arcsin \frac{1}{2} = 15°.$$

The corresponding sequence is

$$2,\ 4,\ 14,\ 52,\ 194, \ldots.$$

22.5°

Take

$$M = 6, \qquad k = 2.$$

Equation (7) gives

$$\tan\theta_{6,2} = \exp\left(-\frac{1}{2}\operatorname{arcosh} 3\right).$$

Since

$$\operatorname{arcosh} 3 = \log(3 + 2\sqrt{2}) = 2\log(1 + \sqrt{2}),$$

we obtain

$$\tan\theta_{6,2} = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1.$$

Therefore

$$\theta_{6,2} = 22.5^\circ.$$

Its sequence is

$$2,\ 6,\ 34,\ 198,\ 1154,\ 6726, \ldots.$$

Thus

$$15^\circ \longleftrightarrow (M, k) = (4, 1),$$

while

$$22.5^\circ \longleftrightarrow (M, k) = (6, 2).$$

7 Many angles, one sequence

A striking feature of the construction is that the recurrence

$$A_n = MA_{n-1} - A_{n-2}$$

contains no k.

Thus, for fixed $M > 2$, different positive integers k give different angles $\theta_{M,k}$, but all of them generate exactly the same integer sequence.

For example,

$$\theta_{4,1} = 15^\circ,$$

whereas

$$\theta_{4,2} = \arctan\left(e^{-\frac{1}{2}\operatorname{arcosh} 2}\right) \approx 27.368^\circ.$$

The angles differ, yet

$$\tan^n 15^\circ + \cot^n 15^\circ$$

and

$$\tan^{2n}\theta_{4,2} + \cot^{2n}\theta_{4,2}$$

both produce

$$2,\ 4,\ 14,\ 52,\ 194, \ldots.$$

This independence can also be seen directly. Put

$$\alpha = \frac{M + \sqrt{M^2 - 4}}{2}.$$

From (7), $\tan\theta_{M,k} = \alpha^{-1/k}$, so

$$\tan^{kn}\theta_{M,k} + \cot^{kn}\theta_{M,k} = \alpha^{-n} + \alpha^{n},$$

which depends only on M and n, not on k.

Figure 1 shows the first few $M = 4$ realizations. For example, $k = 1, 2, 3, 4$ give approximately $15.000°, 27.368°, 32.809°, 35.734°$, respectively, and all generate the same sequence $2, 4, 14, 52, 194, \ldots$. Moreover, for fixed $M > 2$, we have $\alpha > 1$, so $\alpha^{-1/k}$ is strictly increasing with k. Hence

$$\theta_{M,1} < \theta_{M,2} < \theta_{M,3} < \cdots < 45°, \qquad \theta_{M,k} \longrightarrow 45°.$$

The case $M = 2$ is degenerate: equation (8) gives

$$\theta_{2,k} = 45°$$

for every k, and the resulting sequence is simply

$$2,\ 2,\ 2, \ldots.$$

So, except for this boundary case, a single integer sequence has infinitely many distinct trigonometric realizations.

8 Lucas sequences

Recurrences of this form belong to the classical theory of Lucas sequences, and their connection with Pell equations is well known; see, for example, Lucas [1] and Ayoub [4]. Our purpose here is different: starting from symmetric trigonometric powers, we obtain infinitely many trigonometric realizations of the same recurrence.

The name also carries some useful history. Edouard Lucas (1842–1891) worked extensively with Fibonacci-type recurrences and with tests for primality, especially for Mersenne numbers. His investigations of these recurrences were later extended by D. H. Lehmer [5]. In the present setting, the Lucas connection emerges directly from the two quadratic conjugates hidden in the trigonometric powers.

More generally, the Lucas sequence $V_n(P, Q)$ is defined by

$$V_0 = 2, \qquad V_1 = P, \qquad V_n = PV_{n-1} - QV_{n-2}.$$

The roots of

$$x^2 - Mx + 1 = 0$$

are

$$\alpha = \frac{M + \sqrt{M^2 - 4}}{2}, \qquad \beta = \frac{M - \sqrt{M^2 - 4}}{2}.$$

Since

$$\alpha\beta = 1,$$

equation (6) has the closed form

$$A_n = \alpha^n + \beta^n = \left(\frac{M + \sqrt{M^2 - 4}}{2}\right)^n + \left(\frac{M - \sqrt{M^2 - 4}}{2}\right)^n. \tag{10}$$

This is the Lucas sequence

$$A_n = V_n(M, 1).$$

It also has the Chebyshev representation

$$A_n = 2T_n(M/2). \tag{11}$$

Chebyshev polynomials have therefore appeared twice:

$$\tan^k \theta + \cot^k \theta = 2T_k\left(\frac{\tan\theta + \cot\theta}{2}\right),$$

and

$$A_n = 2T_n(M/2).$$

The same polynomial structure governs both the power index k and the sequence index n.

A Lucas–Lehmer surprise

For the opening example $M = 4$, we have $\alpha = 2 + \sqrt{3}$ and $\beta = 2 - \sqrt{3}$, with $\alpha\beta = 1$. Therefore

$$A_{2n} = \alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2(\alpha\beta)^n = A_n^2 - 2.$$

If we select only the power-of-two indices and write

$$s_m = A_{2^m},$$

then

$$s_{m+1} = s_m^2 - 2, \qquad s_0 = 4,$$

so

$$4,\ 14,\ 194,\ 37634, \ldots$$

appears as a subsequence of the original 15° sequence. This is precisely the Lucas–Lehmer sequence used in the classical primality test for Mersenne numbers [6]. In the present trigonometric notation,

$$\boxed{s_m = \tan^{2^m} 15^\circ + \cot^{2^m} 15^\circ}.$$

Figure 2 shows how repeated index doubling extracts this nonlinear recurrence from the linear Lucas sequence.

9 Pell-type equations

There is a small historical twist here. Equations of the form $x^2 - Dy^2 = 1$ were studied in India centuries before their appearance in European mathematics. In 1657 Fermat posed equations of this type as challenge problems; Brouncker developed a method of solution, and Euler later attached Pell's name to the equation. Lagrange subsequently gave the general continued-fraction theory. A concise account of this history is given by Barbeau [3].

Assume now that $M > 2$. Define the companion sequence

$$B_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{12}$$

It satisfies

$$B_0 = 0, \qquad B_1 = 1,$$

and the same recurrence

$$B_n = MB_{n-1} - B_{n-2}.$$

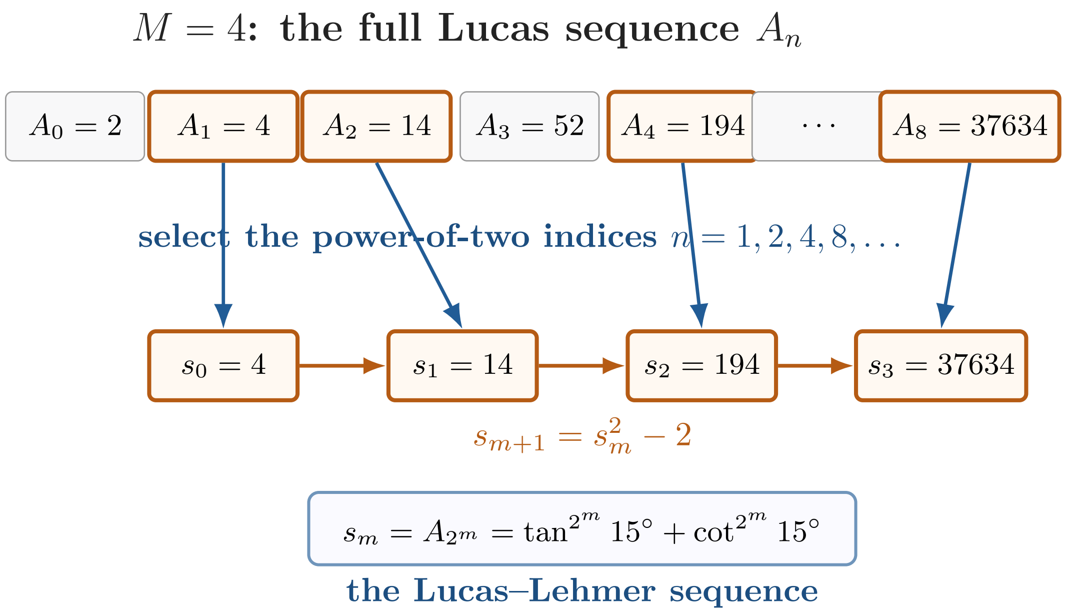

Figure 2: The Lucas–Lehmer sequence is the power-of-two subsequence A_{2^m} of the $M = 4$ Lucas sequence. The identity $A_{2n} = A_n^2 - 2$ turns index doubling into the nonlinear recurrence $s_{m+1} = s_m^2 - 2$.

Since

$$\alpha - \beta = \sqrt{M^2 - 4},$$

we have

$$\alpha^n - \beta^n = B_n \sqrt{M^2 - 4}.$$

Therefore

$$\begin{aligned} A_n^2 - (M^2 - 4)B_n^2 &= (\alpha^n + \beta^n)^2 - (\alpha^n - \beta^n)^2 \\ &= 4(\alpha\beta)^n \\ &= 4. \end{aligned}$$

Hence

$$A_n^2 - (M^2 - 4)B_n^2 = 4. \tag{13}$$

This is a Pell-type equation; for background, see Barbeau [3].

For $M = 4$,

$$A_n^2 - 12B_n^2 = 4.$$

Since A_n is even, put

$$A_n = 2X_n.$$

Then

$$X_n^2 - 3B_n^2 = 1. \tag{14}$$

The first few solutions are

$$(X_n, B_n) = (2, 1),\ (7, 4),\ (26, 15),\ (97, 56), \ldots.$$

For $M = 6$,

$$A_n^2 - 32B_n^2 = 4.$$

Writing

$$A_n = 2X_n, \qquad Y_n = 2B_n,$$

gives

$$X_n^2 - 2Y_n^2 = 1. \tag{15}$$

Thus the 15° and 22.5° examples lead respectively to Pell equations associated with $\sqrt{3}$ and $\sqrt{2}$.

10 One structure, two recurrences

The initial puzzle reveals two recurrence directions. For a fixed angle, $\tan^k\theta + \cot^k\theta$ varies with k according to a Chebyshev recurrence. Once one such value is an integer M, fixing that power and varying n gives the Lucas recurrence

$$A_n = MA_{n-1} - A_{n-2}.$$

For fixed $M > 2$, infinitely many distinct choices of angle and exponent lead to exactly the same Lucas sequence; the boundary case $M = 2$ reduces to 45° and the constant sequence $2, 2, 2, \ldots$. The companion sequence then brings Pell-type equations into the same picture.

The route is therefore

$$\boxed{\text{angle} \longrightarrow \text{symmetric powers} \longrightarrow \text{Chebyshev} \longrightarrow \text{Lucas} \longrightarrow \text{Pell.}}$$

The surprising feature is that this route is many-to-one: infinitely many choices of angle and exponent can lead to exactly the same Lucas sequence.

References

[1] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math. **1** (1878), 184–196, 197–240, 289–321.

[2] T. J. Rivlin, *Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory*, 2nd ed., Wiley, New York, 1990.

[3] E. J. Barbeau, *Pell's Equation*, Springer, New York, 2003.

[4] A. B. Ayoub, Fibonacci-like sequences and Pell equations, *College Math. J.* **38** (2007), 49–53.

[5] D. H. Lehmer, An extended theory of Lucas' functions, *Ann. of Math.* **31** (1930), 419–448.

[6] D. H. Lehmer, On Lucas's test for the primality of Mersenne's numbers, *J. London Math. Soc.* **10** (1935), 162–165.