Visualising Fermat's Last Theorem with Proofs Within Infinite Families of Pythagorean Triples

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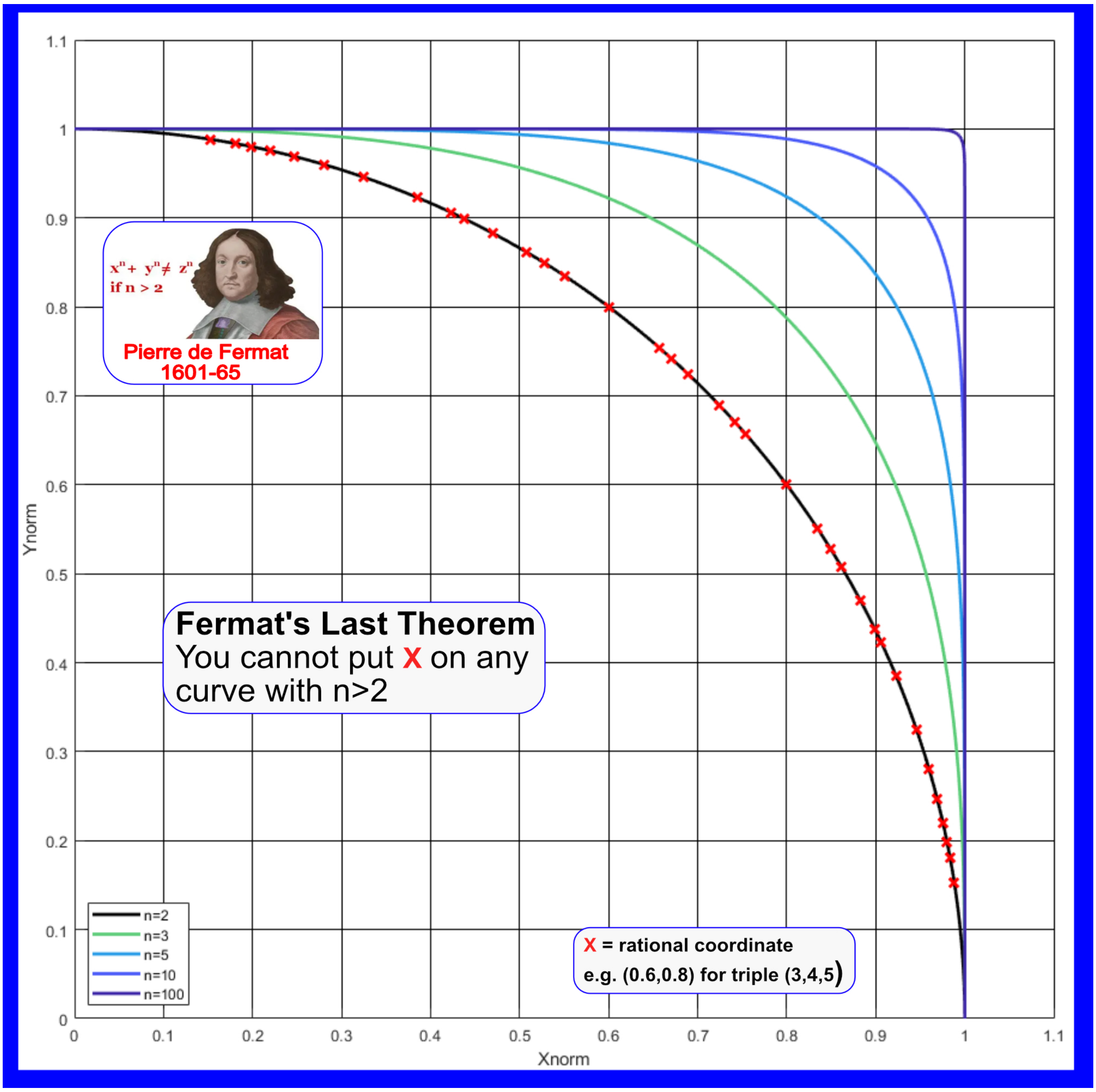
Abstract

Fermat's Last Theorem asserts that there are no non-zero integers x, y, z satisfying $x^n + y^n = z^n$ for any integer $n > 2$. Although Professor Wiles' celebrated proof establishes this result using deep methods from arithmetic geometry (Appendix A), its technical nature places it beyond the reach of most readers.
This work develops an elementary and visual geometric framework for understanding why Fermat-type equations admit no integer solutions in certain structured settings. The approach centres on *succession-t triples* - primitive Pythagorean triples with a fixed hypotenuse gap - and their associated *normalised Fermat plots.* For each fixed t, we show that only finitely many triples require explicit checking; beyond this finite threshold, all rational directions determined by succession-t triples intersect the Fermat curves $x^p + y^p = 1$ (for $p > 2$) only at irrational points.
This provides rigorous proofs of Fermat's equation within infinite families of Pythagorean triples, offering a visual and accessible complement to the classical theory and illustrating how geometric structure can illuminate special cases of Fermat's Last Theorem.

Fermat's Last Theorem Graphic



Summary of the Method of Proof for Selective Pythagorean Triples

Referring to Figure 1, a geometrical construction relates the coordinates (a', b') of points on the curves $x^n + y^n = 1$ for $n > 2$ to the rational Pythagorean triple coordinates (a, b) for $n = 2$. In this paper we exhibit special cases of triples, called succession-t triples, for which the corresponding normalised coordinates (a, b) are rational but the associated factor $(a^n + b^n)^{1/n}$ is shown to be irrational. Since (a', b') is obtained from (a, b) by dividing by this factor, this leads to the irrationality of (a', b') and hence excludes integer Fermat-type solutions in these succession-t cases.

In more detail, we outline the method to prove the irrationality of succession-t triples when projected onto the curves (Lp-norms) for $n > 2$.

- A Pythagorean triple (x, y, z) normalises to $(x/z, y/z, 1)$, giving the rational point $(a, b) = (x/z, y/z)$ on the unit circle with $a^2 + b^2 = 1$.
- Extend the ray from the origin through (a, b) to intersect the Lp-norm curve $x^n + y^n = 1$ at (a', b'), as illustrated for $n = 3$ in Figure 1.
- In Appendix F.2 it is shown that (a', b') is given in terms of (a, b) by

$$a' = \frac{a}{\left(a^n + b^n\right)^{1/n}}, \qquad b' = \frac{b}{\left(a^n + b^n\right)^{1/n}}.$$

Since $a^2 + b^2 = 1$ with $0 < |a|, |b| < 1$ and $n > 2$, we have $a^n + b^n < 1$, so $(a^n + b^n)^{1/n} < 1$ and consequently $|a'| > |a|$, $|b'| > |b|$.

Rational and irrational points

- Rational points on the $n = 2$ curve are coordinates (a, b) such that both a and b are rational.
- Irrational points on the $n > 2$ curves are coordinates (a', b') such that at least one of a' and b' is irrational.

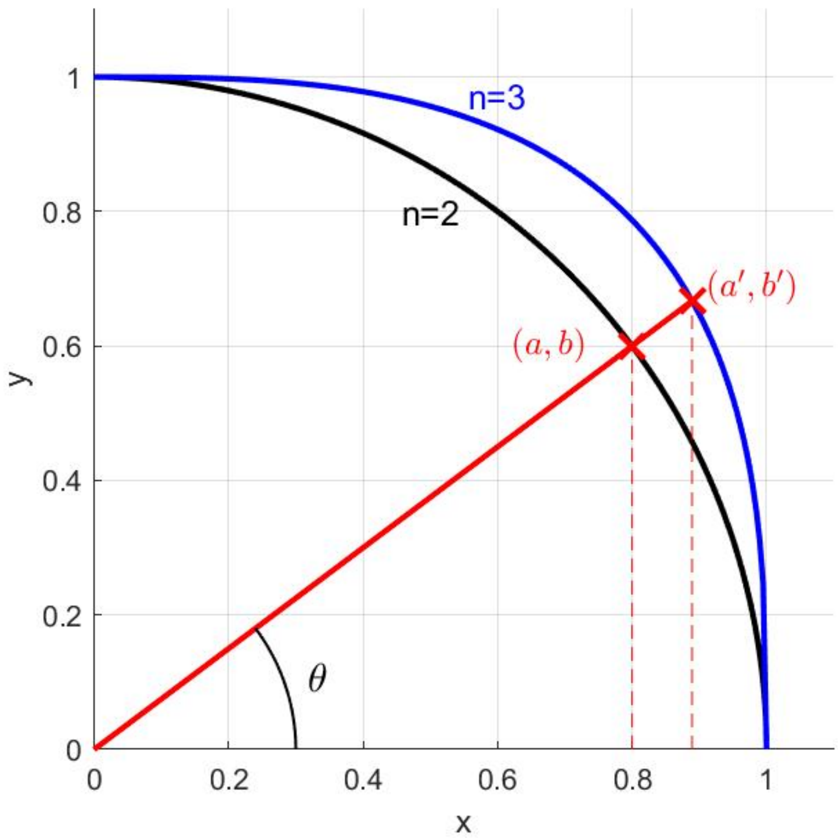


Figure 1: Rational slope intersecting the normalised Fermat curves (L2 and L3-norms).

- For the succession-t special cases considered in this paper, we show that $\left(a^n + b^n\right)^{1/n}$ is irrational. Since a, b are rational and at least one is non-zero, dividing by an irrational quantity forces at least one of a', b' to be irrational. Thus, the intersections of the Lp-norm curves along these rational rays yield irrational points, which excludes corresponding integer solutions to $x^n + y^n = z^n$ in these succession-t cases, proving FLT for these cases.

Introducing Succession-t Pythagorean Triples

What are Succession-t Triples?

These are defined as $x_p^2 + y_p^2 = z_p^2 = (x_p + t)^2$ where t is an integer and p denotes Pythagorean triples, under the constraints that (x_p, y_p, z_p) is primitive, i.e. coprime with $\gcd(x_p, y_p, z_p) = 1$ and $x_p > y_p$. The constraints take into account the symmetry of the curves about $y = x$ and eliminate triple multiples.

Example of Primitive Succession-1 Triples

Here we have $x_p^2 + y_p^2 = (x_p + 1)^2$ giving

$$y_p^2 = 2x_p + 1$$

so that

$$x_p = \frac{\left(k^2 - 1\right)}{2}.$$

These are some solutions: $k = 3$:

$$x_p = \frac{\left(3^2 - 1\right)}{2}$$

giving $x_p = 4$ with $k = y_p = 3$ and $z_p = x_p + 1 = 5$ giving triple $(4, 3, 5)$.
$k = 5$:

$$x_p = \frac{\left(5^2 - 1\right)}{2}$$

giving $x_p = 12$ with $k = y_p = 5$ and $z_p = x_p + 1 = 13$ giving triple $(12, 5, 13)$.

An Infinite Number of Succession-t Triples

Since k and t are integers, there are a countably infinite (Appendix E) number of x_p and y_p $(=k)$, therefore there are a countably infinite number of succession-t triples.

Parity of Succession-t Triples

Appendix G.7 shows, for a triple (a, b, c) coprime and $a > b$:

- If t is odd then a is even, b is odd and c is odd.
- If t is even then a is odd, b is even and c is odd.

This will restrict the number of possible integer solutions available.

Existence of Succession-t Triples

Appendix G.3 shows, for (a, b, c) coprime and $a > b$, not all succession-t triples exist. For example, for $t < 100$ there are only succession triples for the following t: 1,2,8,9,18,25,32,49,50,72,81,98.
Examples of the first three for $t = 18$: $(55, 48, 73)$, $(91, 60, 109)$, $(187, 84, 205)$. Examples of the first three for $t = 81$: $(224, 207, 305)$, $(272, 225, 353)$, $(380, 261, 461)$.

Describing the Visualisation of Fermat's Last Theorem

This visualisation is shown in Figure 2 which is a plot of $z = (x^n + y^n)^{\frac{1}{n}}$ for various n, normalised by z to become $(Xnorm^n + Ynorm^n)^{\frac{1}{n}} = 1, \quad x, y, z \in \mathbb{R}$. Pythagorean triples are the red crosses for $n = 2$. For example, for the triple $(3, 4, 5)$ the rational point is (0.6,0.8). FLT states it is not possible to put rational crosses onto the other curves.

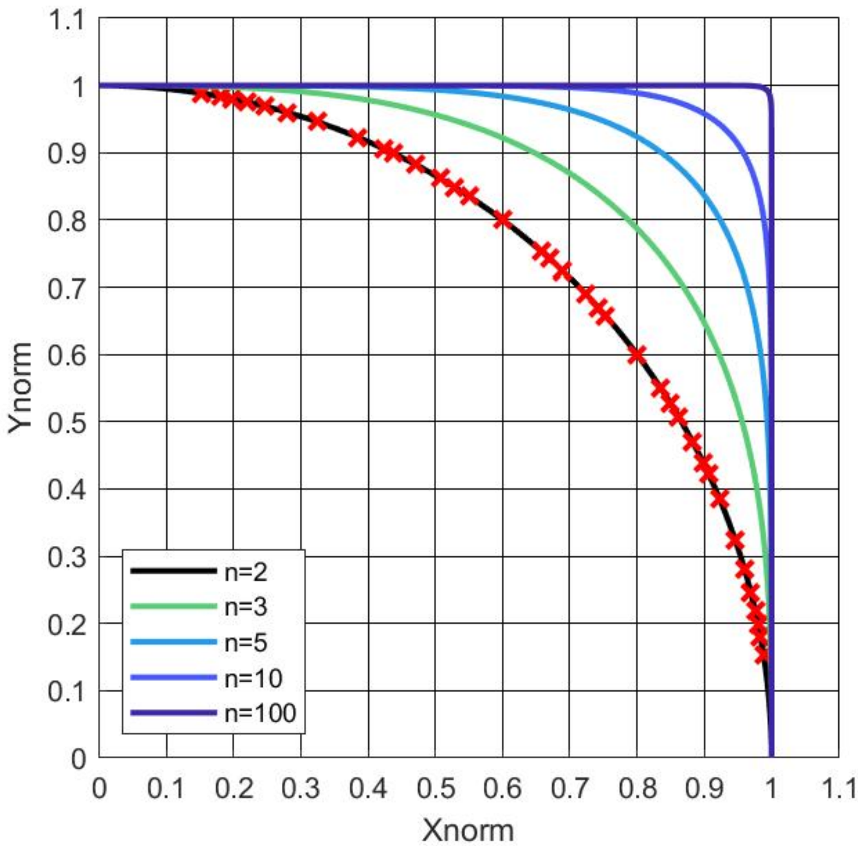


Figure 2: The Visualisation of Fermat's Last Theorem $x^n + y^n = z^n$ for various n.

- These curves are named here normalised Fermat Plots (formally known as Lp-norms)
- For $n = 2$ the curve and markers together are known as the rational points on the unit circle
- The curves are symmetric about $y = x$
- For $n > 3$ the curves are non-rational (informally 'irrational') see Appendix B

p-norm (or Lp-norm) Plots

These curves are formally known as p-norms or more generally Lp-norms, the L named for Lebesgue although generalised by Hungarian mathematician Frigyes Riesz in 1910, where for real p the norm $p \geq 1$ $\|(a, b)\|_p = (|a|^p + |b|^p)^{1/p}$.

3D Fermat Plots

The evolution of the visualisation plot is shown in Figure 3 .
As an example, for $n = 2$ this is a plot of $z = \left(x^2 + y^2\right)^{\frac{1}{2}}$ named here as a 3D Fermat plot.

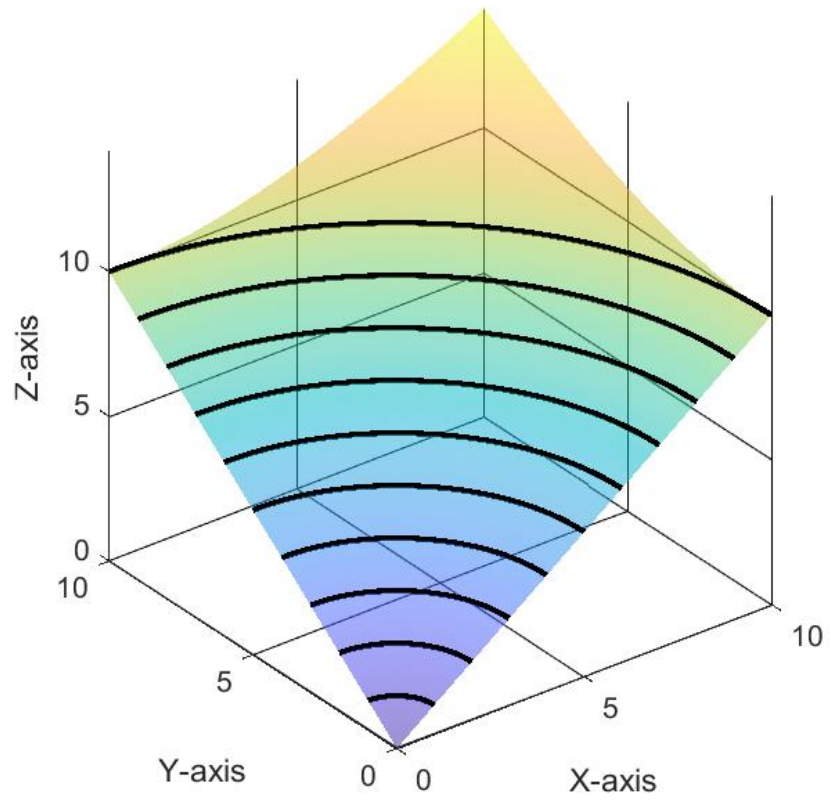


Figure 3: 3D Fermat plot $n = 2$. Unit contours of z are shown.

Integer planar cuts in the z axis are examples of superquadrics. Further development towards the visualisation plot are shown in Appendix C.

Outline of Proof of Irrationality of Selected Succession-t Triples

- Normalising a Pythagorean triple (x, y, z) gives

$$(a, b, 1) = \left(\frac{x}{z}, \frac{y}{z}, 1\right),$$

  so $a^2 + b^2 = 1$ with $a, b \in \mathbb{Q}$ and $0 < |a|, |b| < 1$ for a non-trivial triple. For any integer $n > 2$ and $0 < |u| < 1$ we have $|u|^n < |u|^2$, hence $a^n + b^n < a^2 + b^2 = 1$, that is $a^n + b^n < 1$. Multiplying by z^n shows $x^n + y^n < z^n$; for example, for the succession-1 triple $(4, 3, 5)$ we have $4^3 + 3^3 = 91 < 5^3 = 125$ (Appendix D).
- In general, we will be looking for odd integer (Appendix G.7) solutions for

$$c' = c^{1/n}$$

  with $x < c' < x + t$, where $c = x^n + y^n$ and $t = z - x$. A basic number-theoretic fact is that if $c^{1/n}$ is an integer then c must be a perfect nth power (Niven [1]); otherwise c' is irrational. For example, since 91 is not a perfect cube, $c' = 91^{1/3}$ is the irrational number $4.4797\ldots$. In this example with $t = 1$ there is no integer between 4 and 5, and more generally between x and $z = x + 1$, so there is no integer solution to the succession-1 triple

$$x^3 + y^3 = z^3 = (x + 1)^3,$$

  or in general to $x^n + y^n = z^n = (x + 1)^n$.
- Since there are countably infinitely many succession-1 triples, applying the geometrical construction of Appendix F.2 to each such triple yields a countably infinite family of irrational coordinate intersections on the Lp-norm curves.
- We will then computationally extend this analysis to succession-t triples for t up to 8 and open the discussion for larger t.

Proof of the Irrationality for Succession-1 Triples

Consider the succession-1 Pythagorean triple (x_p, y_p, z_p) so that $z_p = x_p + 1$, giving $(x_p, y_p, x_p + 1)$, e.g. $(4, 3, 5)$. Normalising to z_p gives $\left(\frac{x_p}{z_p}, \frac{x_p}{z_p}, 1\right)$, and we label these normalised rational Pythagorean triples (a_p, b_p), e.g. $(0.8, 0.6)$ for triple $(4, 3, 5)$. By definition $a_p^2 + b_p^2 = 1$, and since $a_p < 1$, $b_p < 1$, raising to a higher power yields $a_p^n + b_p^n < 1$ for $n > 2$, e.g. $0.8^3 + 0.6^3 < 1$, or unnormalised $4^3 + 3^3 < 5^3$.

For a succession-1 triple $(x_p, y_p, x_p + 1)$ with $n > 2$,

$$x_p^n \ < \ x_p^n + y_p^n \ < \ (x_p + 1)^n.$$

Taking nth roots (monotone for positive numbers),

$$x_p \ < \ (x_p^n + y_p^n)^{1/n} \ < \ x_p + 1.$$

Since there is no integer strictly between x_p and $x_p + 1$, $(x_p^n + y_p^n)^{1/n}$ cannot be an integer.
A positive integer nth root of an integer is either an integer or irrational, so $(x_p^n + y_p^n)^{1/n}$ and its normalisation $(a_p^n + b_p^n)^{1/n}$ must be irrational.

Substituting these irrational succession-1 values into the equations from the geometrical construction of Appendix F.2,

$$a' = \frac{a_p}{(a_p^n + b_p^n)^{1/n}}, \qquad b' = \frac{b_p}{(a_p^n + b_p^n)^{1/n}},$$

and since by definition a_p and b_p are non-zero rationals and since $(a_p^n + b_p^n)^{1/n}$ is irrational for all $n > 2$, therefore (a'_p, b'_p) is an irrational coordinate. Therefore FLT is proved for a countably infinite number of succession-1 triples.

Proof of Irrationality for Succession-2 Triples

Using the same reasoning as for succession–1 triples, we must show that for a succession–2 triple $(x_p, y_p, x_p + 2)$ with $n > 2$,

$$x_p \ < \ (x_p^n + y_p^n)^{1/n} \ < \ x_p + 2.$$

Since the nth root is strictly between two consecutive even integers, the *only* possible integer value it could take is the midpoint $x_p + 1$.

Figure 4 lists the first three succession–2 triples:

$$(15, 8, 17), \qquad (35, 12, 37), \qquad (63, 16, 65),$$

indexed by $s = 1, 2, 3$. For each triple, the only possible integer candidate between x_p and $x_p + 2$ is

$$x_p + 1 = 16, \ 36, \ 64.$$

However, since $t = 2$ is even, the parity analysis of Appendix G.7 shows that any integer solution c' to $(x_p^n + y_p^n)^{1/n}$ must be *odd*. The only available candidates $x_p + 1$ are all *even*. Therefore no integer solution is possible for any $n > 2$ for these triples.

This parity obstruction persists for all succession–2 triples, as further examples in Figure 5 demonstrate. Hence for every succession–2 triple, the value $(x_p^n + y_p^n)^{1/n}$ is irrational for all $n > 2$.

a	**b**	**c**	**c'²**	**c'**	**c'³**	**c'**	**c'⁴**	**c'**
15	8	17	289	**17**	3887	**15.723**	54721	**15.295**
35	12	37	1369	**37**	44603	**35.464**	1521361	**35.120**
63	16	65	4225	**65**	254143	**63.342**	15818497	**63.065**

Figure 4: Higher-power values for the first three succession–2 triples. In each case, the only possible integer candidate is even, while parity requires any solution to be odd. Thus no integer solutions exist for $n > 2$.

a	b	c	Possible Integer Odd Solutions c'
15	8	17	None
35	12	37	None
63	16	65	None
143	24	145	None
195	28	197	None
255	32	257	None

Figure 5: This shows, for example, there are no possible integer solutions for the first six succession-2 triples.

Succession-3 to 7 Triples

Appendix G.3 shows that, under the constraints of primitivity and the ordering $x > y$, there are *no* primitive succession-t triples for $t = 3, 4, 5, 6, 7$. These values of t are ruled out by the Euclid parameterisation, which forces

$$t = 2s^2 \quad \text{or} \quad t = (r - s)^2,$$

together with the parity and coprimality conditions on (r, s) and the inequalities required by $x > y$. Since none of the integers $3, 4, 5, 6, 7$ can be written in either of these forms while also satisfying the parity and gcd restrictions, no primitive triples of these types exist.

In particular, any hypothetical succession-t triple with $3 \leq t \leq 7$ would have to arise from one of the two Euclid assignments already used to generate the valid succession-t families. But the only admissible values of t produced by these assignments are those appearing in the existing families (e.g. $t = 1, 2, 8, 9, 18, \ldots$). Thus the cases $t = 3$ to $t = 7$ introduce no new behaviour: they are already implicitly accounted for by the structural constraints of the Euclid parameterisation. In this sense, the "missing" succession-t triples for $3 \leq t \leq 7$ are not new families at all, but simply values of t that cannot occur within the primitive Pythagorean framework.

Proof of Irrationality for Succession-8 Triples

For a succession–8 triple $(x_p, y_p, x_p + 8)$ and any exponent $n > 2$, the same interval argument applies as in the cases $t = 1$ and $t = 2$. Since $y_p > 0$, we have

$$x_p^n \; < \; x_p^n + y_p^n \; < \; (x_p + 8)^n,$$

and taking nth roots (monotone for positive numbers) gives

$$x_p \; < \; (x_p^n + y_p^n)^{1/n} \; < \; x_p + 8.$$

Thus any possible integer value of $(x_p^n + y_p^n)^{1/n}$ must lie among the integers

$$x_p + 1, \; x_p + 2, \; \ldots, \; x_p + 7.$$

Figure 6 lists the first six succession–8 triples and the corresponding values of

$$c' = (x_p^n + y_p^n)^{1/n}$$

for exponents up to $k = 7$. For each triple, the lowest possible *odd* integer candidate is $x_p + 1$, since $t = 8$ is even and the parity analysis of Appendix G.7 shows that any integer solution c' must be odd.
As an example, for the first succession–8 triple $(21, 20, 29)$, the lowest possible odd integer candidate is 23. The table shows that

$$21^6 + 20^6 = 149{,}766{,}121 = c'^6,$$

so

$$c' = (149{,}766{,}121)^{1/6} = 23.04\ldots,$$

which is *not* an integer. This is the largest exponent k for which c' lies above the lowest possible odd integer candidate. For the next exponent $k = 7$, we obtain $c' = 22.67\ldots$, which is already below 23. Hence, for the first triple, it is sufficient to check exponents only up to $k = 6$.

A similar pattern holds for the next three triples: for $s = 2, 3, 4$ it is sufficient to check up to $k = 3$, since beyond this point the values of c' fall below the lowest possible odd integer candidate. For the fifth and subsequent triples, no checking is required at all, because the lowest possible odd integer candidate already exceeds the maximum possible value of c' for all $k \geq 2$.

a	b	c	c'^3	**c'**	c'^4	**c'**	c'^5	**c'**	c'^6	**c'**	c'^7	**c'**
21	20	29	17261	**25.84**	354481	**24.40**	7284101	**23.58**	149766121	**23.04**	3081088541	**22.67**
45	28	53	113077	**48.36**	4715281	**46.60**	201738493	**45.81**	8785655929	**45.43**	3.87162E+11	**45.23**
77	36	85	503189	**79.54**	36832657	**77.90**	2767250333	**77.34**	2.10599E+11	**77.13**	1.61269E+13	**77.05**
117	44	125	1686797	**119.04**	191136817	**117.58**	22089396581	**117.18**	2.57242E+12	**117.06**	3.00443E+14	**117.02**
165	52	173	4632733	**166.70**	748512241	**165.41**	1.22678E+11	**165.10**	2.0199E+13	**165.03**	3.33059E+15	**165.01**
221	60	229	11009861	**222.46**	2398403281	**221.30**	5.27961E+11	**221.07**	1.16554E+14	**221.01**	2.57509E+16	**221.00**

Figure 6: Values of $c' = (x_p^n + y_p^n)^{1/n}$ for the first six succession–8 triples and exponents up to $k = 7$. In each case, c' never attains an odd integer value for $n > 2$.

These results are summarised in Figure 7. The quantity $Ltmax$ denotes the real exponent k for which $c' = x_p + 1$, i.e. the lowest possible odd integer candidate. For the first triple ($s = 1$), Ltmax lies between 6 and 7; for the second triple ($s = 2$), Ltmax lies between 3 and 4. Ltmax is a function of the index s (equivalently b) and of the succession parameter t.

Triple number							**c' for k=**					
s	**a**	**b**	**c**	Possible Odd Integer Solutions c'	**Notes on Hand Checking**	**L_{tmax}**	**6**	**3**	**3**	**3**	**3**	**3**
1	21	20	29	23, 25, 27	Sufficient to check up to c'^6	6.103	23.04					
2	45	28	53	47, 49, 51	Sufficient to check up to c'^3	3.686		48.36				
3	77	36	85	79,81, 83	Sufficient to check up to c'^3	3.224			79.54			
4=Smax	117	44	125	119, 121, 123	Sufficient to check up to c'^3	3.014				119.04		
5	165	52	173	167, 169, 171	No check required - no integer solutions for powers greater than 2	2.892					166.70	
6	221	60	229	223, 225, 227	No check required - no integer solutions for powers greater than 2	2.809						222.46

Figure 7: Maximum exponent k requiring manual checking for the first six succession–8 triples. For $s \geq 5$, Ltmax falls below 3, so no integer solutions are possible for any $n > 2$.

Succession-8 Triples: Geometric Illustration on the Lp-norm Curves

Figure 8 provides a geometric confirmation of the algebraic results proved above. The figure shows the lines from the origin through the first four normalised rational succession–8 triples

$$(0.72, 0.69),\ (0.85, 0.53),\ (0.91, 0.42),\ (0.94, 0.35),$$

marked with red crosses. Each line is extended to intersect the Lp-norm curves for $p = 3, 4, 5, 7, 10$.

As established in the previous section, the corresponding values $(x_p^n + y_p^n)^{1/n}$ are irrational for all $n > 2$. Consistent with this, the figure shows that none of the rational points on these lines lie on any of the Lp-norm curves for $p > 2$. Thus the geometric picture matches the algebraic proof: there are no rational coordinate intersections for any of these succession–8 triples, and therefore no solutions to Fermat's equation along these rational directions.

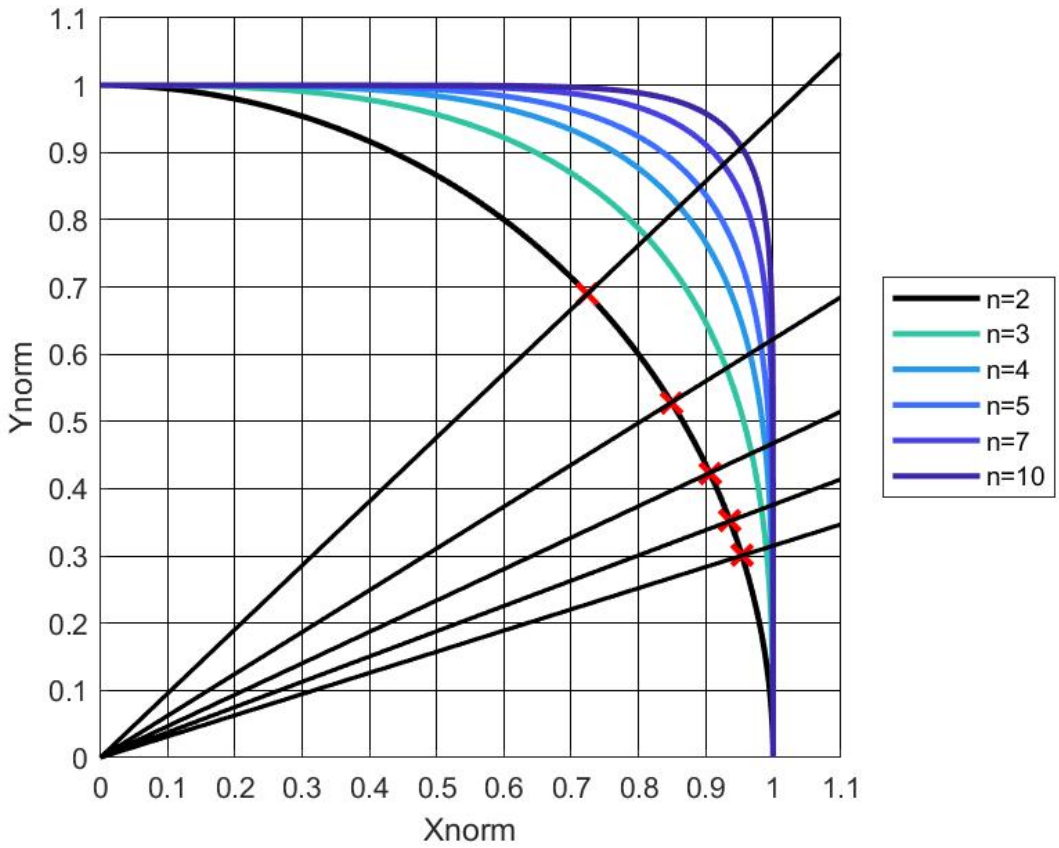


Figure 8: Intersections of the first four succession–8 triples. No rational points (red crosses) lie on any Lp-norm curve for $p > 2$.

How Ltmax Predicts Irrationality Across All Succession-t Triples

Ltmax represents the largest exponent that must be checked by hand for a given succession-t triple. For example, for succession–8 triples we must check up to $k = 6$ for the first triple, up to $k = 3$ for the next three triples, and thereafter no checks are required because Ltmax has fallen below 2. We define the index of the last triple requiring any checking as $S_{\max}$.

More precisely, let $S_{\max}$ be the largest serial number of a triple for which Ltmax ≥ 3, i.e. the final triple before Ltmax drops below 3. For succession–8 triples this occurs at $S_{\max} = 4$.

This behaviour is not unique to $t = 8$. For larger succession values we find, for example,

$$t = 9, 18, 25, 32 \quad \Rightarrow \quad S_{\max} = 9, 10, 61, 38.$$

Thus, for $t = 32$ it is sufficient to check only the first 38 triples by hand; beyond this point Ltmax is already below 2 and no integer solutions are possible. The same pattern continues for all larger t: as t increases, the number of triples requiring manual checking remains finite.

As b (or a) and the exponent k increase, the value $c' = (x_p^n + y_p^n)^{1/n}$ tends monotonically to a. Consequently, the range of possible integer values for c' shrinks rapidly, and fewer triples require manual elimination. For succession–8 triples, only the first four triples require checking (up to exponents $6, 3, 3, 3$ respectively); thereafter no integer solutions are possible.

For succession-1 and succession-2 triples, Ltmax is identically 2, so every triple produces an irrational intersection point on every Lp-norm curve for $p > 2$. Hence there are countably infinitely many such irrational intersections. The same holds for succession-8 triples once the first four triples have been checked, and the method extends directly to all larger t.

This shows: The Ltmax method provides a unified way to prove that, for any succession-t family, only finitely many triples require checking by hand. Beyond $S_{\max}(t)$, all triples yield irrational intersections with the Lp-norm curves for $p > 2$, and therefore no integer solutions to $x^n + y^n = z^n$ exist along those rational directions.

The behaviour of Ltmax and Smax observed for $t = 1, 2, 8$ extends naturally to all succession-t families. The following two subsections summarise these general patterns.

Variation of Ltmax with b

Appendix I shows that Ltmax(b) decreases monotonically and tends asymptotically to 2 as b increases. Thus, for any fixed succession-t family, only finitely many triples require manual checking; beyond this point all triples automatically yield irrational intersections.

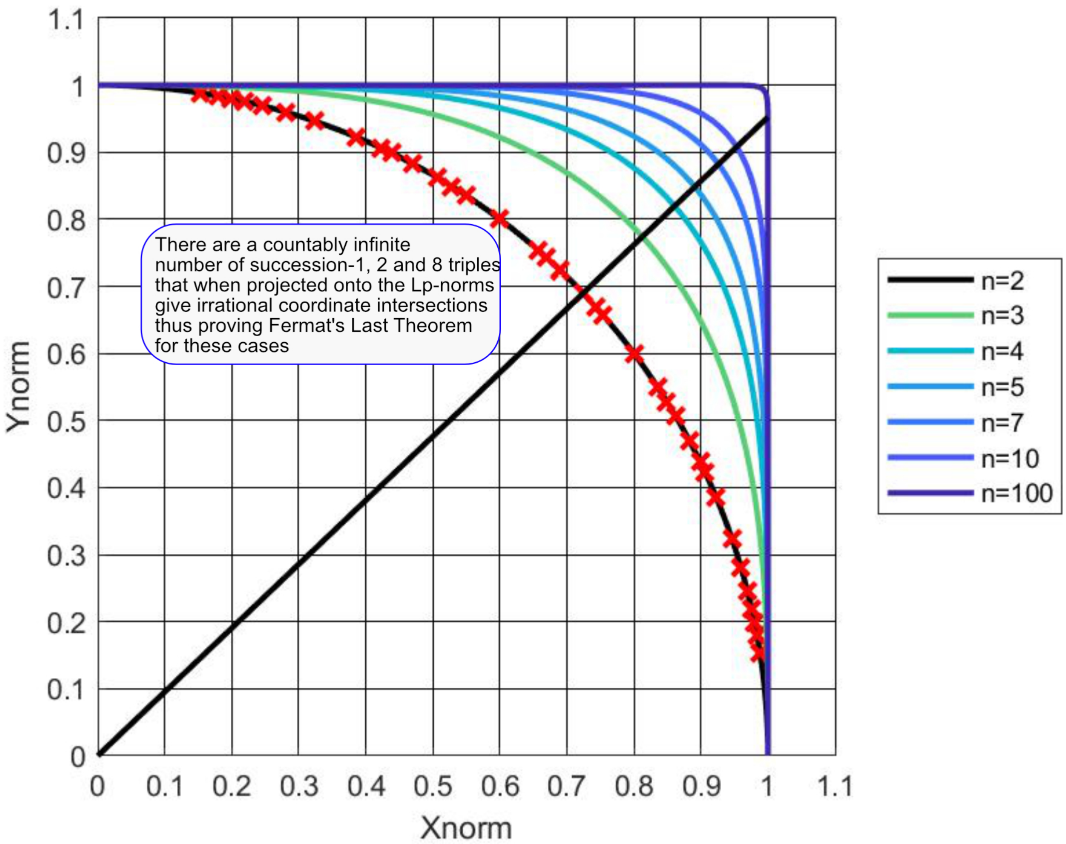
Variation of Smax with t

Appendix J shows that $S_{\max}(t)$ grows in a characteristic sawtooth pattern as t increases. For example, for a succession–968 triple such as $(2765, 2508, 3733)$, one must check the first 6429 triples by hand; all subsequent triples are guaranteed to have irrational intersections. Thus even for very large t, the number of required checks remains finite.

Possible Extensions Beyond Succession-t Families

The results in this paper focus on triples that share a fixed gap between the hypotenuse and one leg. This restriction makes the behaviour of these triples very regular, which is why strong finiteness results can be proved. For general primitive Pythagorean triples, the two parameters that generate them can vary much more freely, producing a wider range of shapes and behaviours. Extending the ideas developed here to all triples would involve studying how these different shapes influence the value of Lt(a, b), how often large exponents occur, and how the corresponding directions meet Fermat curves. A fuller discussion of these possible extensions is provided in Appendix K.

Summary Graphic



Appendix A Skeleton Outline of Professor Wiles' Proof

- **Taniyama–Shimura Conjecture (1950s–1980s):** Every elliptic curve over $\mathbb{Q}$ is modular.
- **Frey's Observation (1984):** A non-trivial solution of

$$x^n + y^n = z^n \quad (n > 2)$$

  would yield a *Frey curve* with abnormal properties, suggesting it is non-modular.
- **Ribet's Theorem (1986):** Ribet showed that if the Frey curve is non-modular, then it contradicts the Taniyama–Shimura Conjecture.
- **Logical Implication:** Thus, if all elliptic curves are modular as predicted by Taniyama–Shimura, no Frey curve can exist; hence, there is no non-trivial solution to Fermat's equation.
- **Professor Wiles' Breakthrough (1993/1995):** Wiles (with Taylor) proved the Taniyama–Shimura Conjecture for semi-stable elliptic curves, thereby confirming Fermat's Last Theorem.

Appendix B Rational and Irrational (Non-Rational) Curves

The Normalised Fermat curves in Figure 2 are defined by

$$a^n + b^n = 1, \quad 0 < a, b < 1,$$

Case $n = 2$: The equation

$$a^2 + b^2 = 1,$$

defines a circle. Despite the fact that almost every point on the circle has irrational coordinates, the circle is *rational* in the algebraic sense.

Case $n = 3$: The equation

$$a^3 + b^3 = 1,$$

defines a cubic curve and hence, for $n = 3$, the normalised curve is elliptic. In algebraic geometry, one describes such a curve as *non-rational* (or sometimes, informally, as "irrational").

Case $n > 3$: They are not elliptic curves and one describes such curves as *non-rational* ("irrational").

In summary, for $n > 2$, one might loosely refer to curves having predominantly irrational coordinates as "irrational curves".

Appendix C Fermat Plots

Figure 9 below shows 3D Fermat plots for $n = 3, 5, 10$ and 100 for x up to 10.

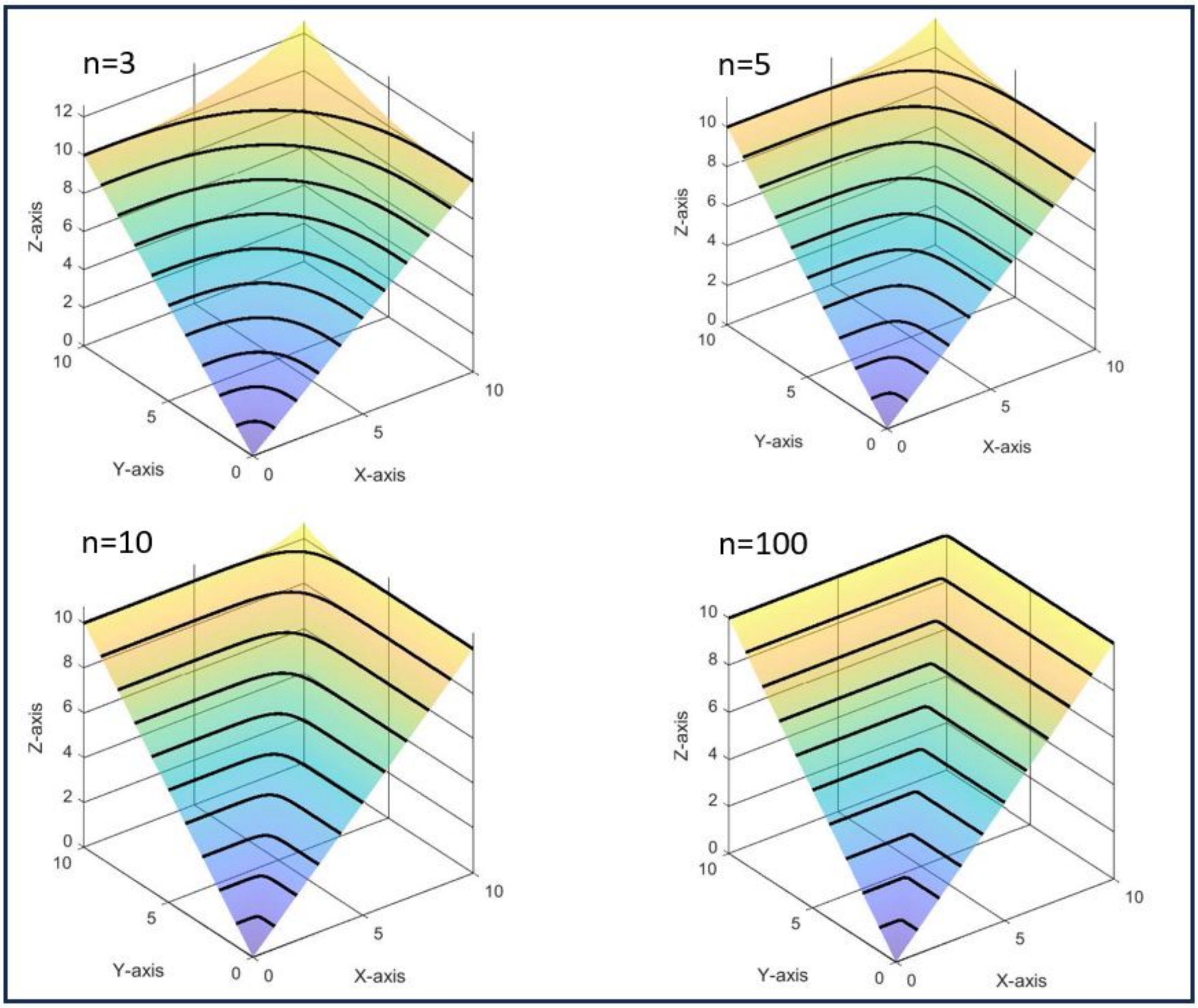


Figure 9: 3D Fermat plots $n = 3, 5, 10, 100$.

Appendix C.1 2D and Normalised Fermat Plot

In Figure 10 the left plot is a plan view of the 3D Fermat plot named a 2D Fermat plot. Since this is for $n = 2$, this shows the Pythagorean triples marked with **x**. Normalising these curves gives the plot on the right of a normalised Fermat plot. This is the same as a unit circle plot with the Pythagorean triples being the rational coordinates on the unit circle.

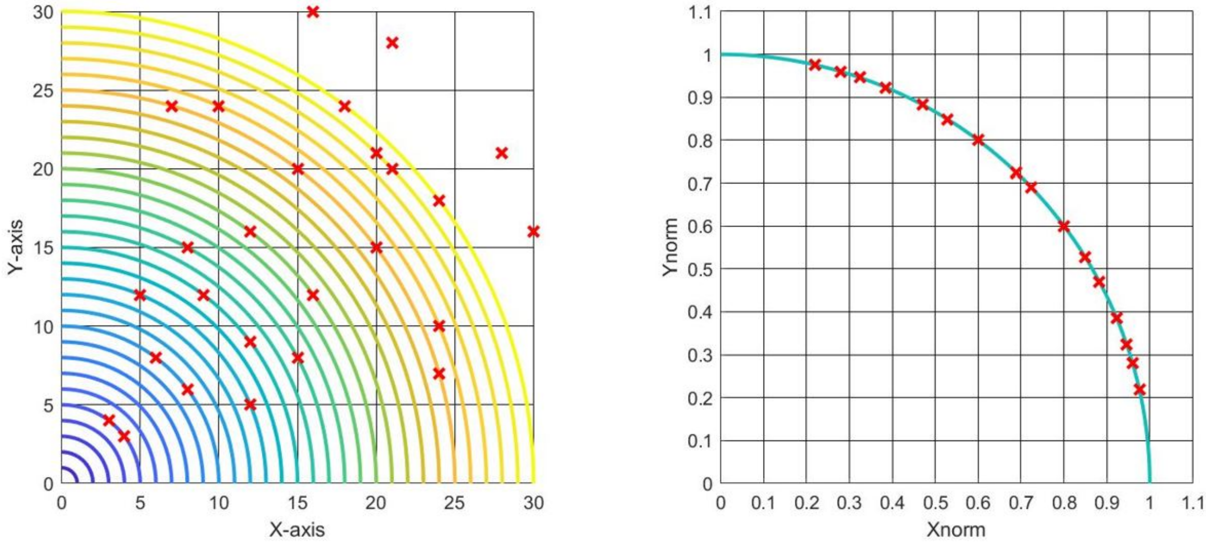


Figure 10: 2D Fermat plot (left) and normalised Fermat plot (right) for $n = 2$ showing Pythagorean triples.

The corresponding Fermat plots for $n = 5$ are shown in Figure 11. Since Fermat's Theorem is true, there are no rational coordinates on this curve.

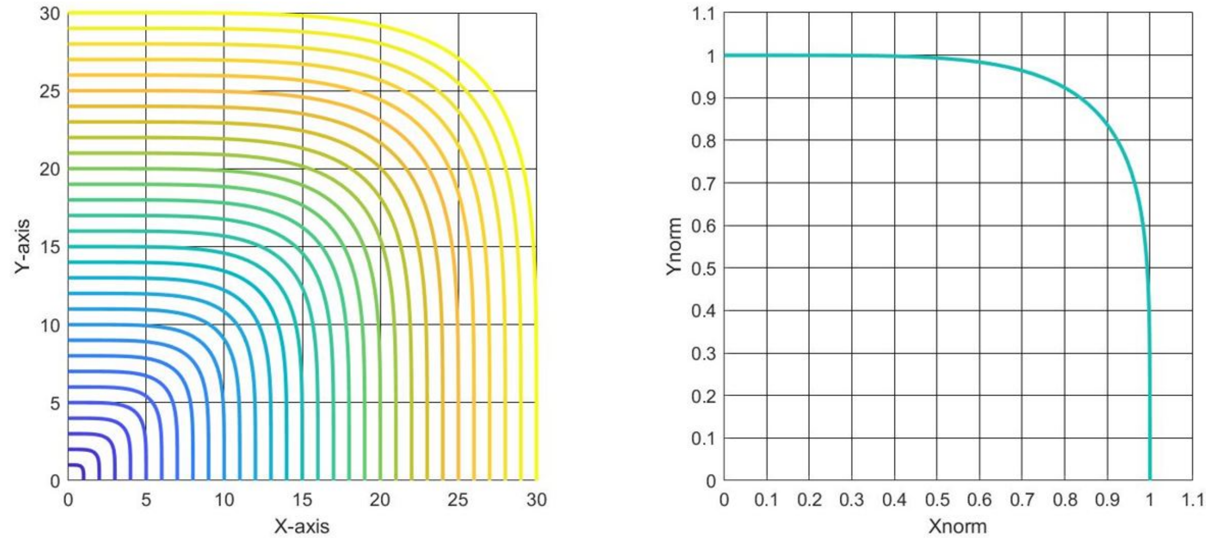


Figure 11: 2D Fermat plot (left) and normalised Fermat plot (right) for $n = 5$.

Appendix D Higher Exponent Normalised Pythagorean Triples

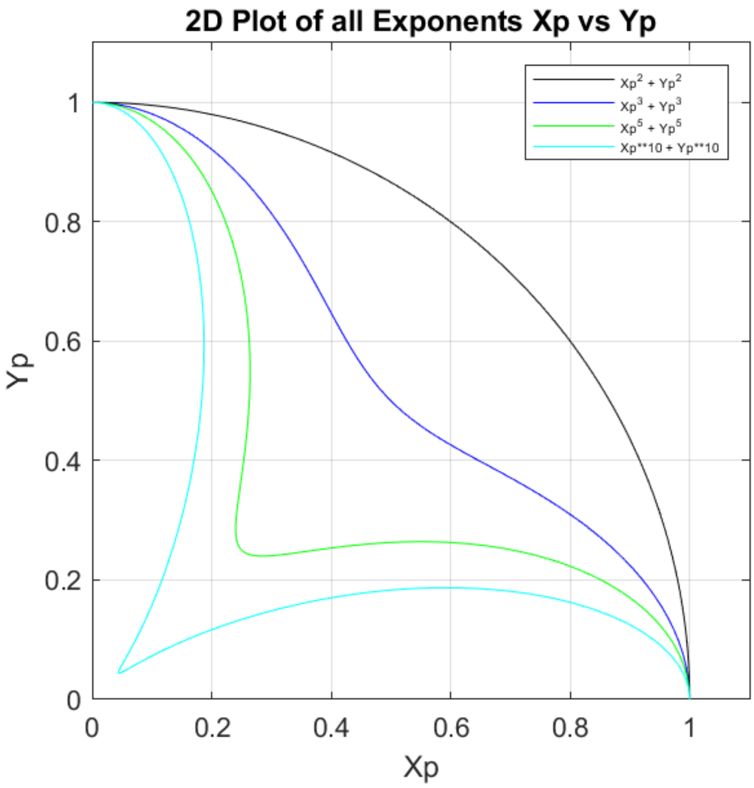


Figure 12: Showing higher exponents of the normalised triples are less than one.

Appendix E Countable and Uncountable Sets

A set is called *countable* if its elements can be placed in one-to-one correspondence with the natural numbers. Typical examples include the integers, the rationals, and any finite set. Even though such sets may be infinite, their elements can still be listed in a sequence.

A set is *uncountable* if no such enumeration is possible. The most famous example is the set of real numbers. In 1874, Georg Cantor (1845–1918) published the first proof that the real numbers are uncountable, and in 1891 he introduced his celebrated *diagonal argument*, showing that any attempted listing of real numbers must necessarily omit some.

Cantor's work, carried out between 1874 and 1897, revealed that infinite sets come in different sizes: the infinity of the real numbers is strictly larger than the infinity of the natural numbers, that is, uncountable and countable infinities.

Appendix F Intersections of the Lp-norm

Appendix F.1 Geometry of the Normalised Fermat Plot

Referring to Figure 13, consider an integer lattice grid in the positive x,y quadrant. We can form rational slopes with vectors from the origin ending on an integer co-ordinate (r, s). In the case of an integer squared hypotenuse, these rational slopes are the Pythagorean triples, with the example $(4, 3, 5)$ shown.

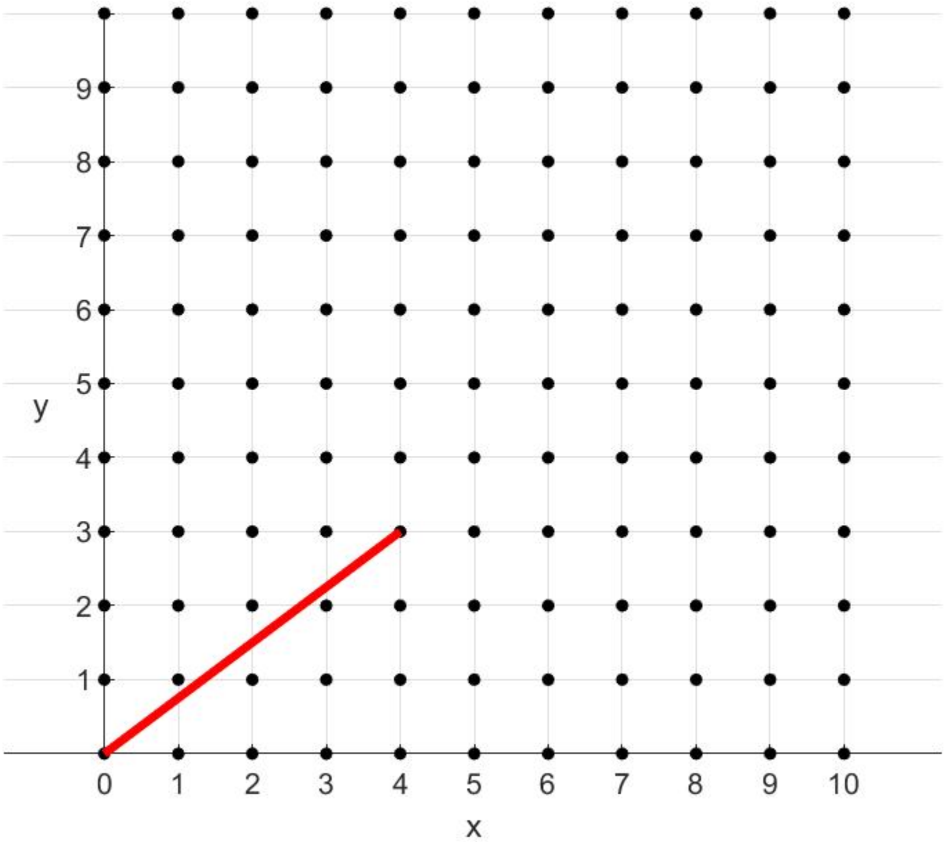


Figure 13: Lattice Points. Example of a rational slope for the triple $(4, 3, 5)$.

A rational slope is shown in Figure 1 intersecting the normalised Fermat plots for $n = 2$ and $n = 3$. The plot is symmetric about $y = x$ and, without loss of generality, the triples for $\theta < 45^\circ$ have $x > y$ as shown for the primitive coprime (x, y, z) triple $(4, 3, 5)$. The triple obtained by reflecting about $y = x$ is $(3, 4, 5)$. This is an example of a Brahmagupta triangle whose side lengths are consecutive positive integers. Extending the rational slope to the normalised curve, for example $n = 3$, gives the coordinate of interception $(a',\, b')$.

Appendix F.2 Geometry of the Intersecting Points

We seek expressions for the normalised coordinates a' and b' in terms of the rational Pythagorean triple a and b only. Since the triangles are similar,

$$\frac{b'}{a'} \;=\; \frac{b}{a} \quad\Longrightarrow\quad b' \;=\; \frac{b}{a}\,a'.$$

Moreover, the normalised curve satisfies

$$(a')^n + (b')^n = 1,$$

so

$$(b')^n = 1 - (a')^n = \frac{b^n}{a^n}\,(a')^n.$$

Putting these together:

$$(a')^n + (a')^n \frac{b^n}{a^n} = 1, \quad (a')^n\Big(1 + \tfrac{b^n}{a^n}\Big) \qquad = 1, \Longrightarrow \quad (a')^n \qquad = \frac{a^n}{a^n + b^n}, \quad a' = \frac{a}{\big(a^n + b^n\big)^{1/n}}.$$

Similarly, from $(b')^n = 1 - (a')^n$ one finds

$$(b')^n = 1 - \frac{a^n}{a^n + b^n} = \frac{b^n}{a^n + b^n} \quad\Longrightarrow\quad b' = \frac{b}{\big(a^n + b^n\big)^{1/n}}.$$

As $n \to \infty$ $a' \to 1$ and $b' \to b/a$ with the slope intersecting $x = 1$.

For any $n > 2$, Fermat's Last Theorem tells us there are no non-zero rational solutions to $a^n + b^n = 1$. It follows that since $(a^n + b^n)^{1/n}$ is irrational, then so are a' and b'. Here we want to apply this to the special cases of succession-t triples.

Appendix F.2.1 How This Applies to the Intersecting Points of Succession-t Triples

If we can show by means independent of Wiles that the denominator of succession-t triples are irrational, which is the subject of this paper, then it follows in the proof below that the projected normalised pair (a', b') onto the Lp-norm curves are necessarily irrational for every exponent $n > 2$, and it follows Fermat's Last Theorem is proved for these cases.

Let

$$a' = \frac{a_p}{(a_p^n + b_p^n)^{1/n}}, \qquad b' = \frac{b_p}{(a_p^n + b_p^n)^{1/n}},$$

where $a_p, b_p \in \mathbb{Q}$ and $(a_p^n + b_p^n)^{1/n} \notin \mathbb{Q}$.

Claim. If $a_p \neq 0$ and $b_p \neq 0$, then a' and b' are irrational.

Proof. Let $q \in \mathbb{Q} \setminus \{0\}$ and let $x \in \mathbb{R} \setminus \mathbb{Q}$. Suppose, for contradiction, that $\frac{q}{x} \in \mathbb{Q}$. Then there exists $r \in \mathbb{Q}$ such that

$$\frac{q}{x} = r \quad \Longrightarrow \quad x = \frac{q}{r},$$

which is a ratio of rationals and therefore rational. This contradicts the assumption that x is irrational. Hence $\frac{q}{x}$ must be irrational.
Applying this with $q = a_p$ or b_p and

$$x = (a_p^n + b_p^n)^{1/n} \notin \mathbb{Q},$$

we conclude that a' and b' are irrational unless the corresponding numerator is zero. If $a_p = 0$ (respectively $b_p = 0$), then $a' = 0$ (respectively $b' = 0$), which is rational.

Appendix G Primitive succession-t triples (x, y, z) with $z = x + t$ and the constraint $x > y$

We analyse integer triples (x, y, z) satisfying

$$x^2 + y^2 = z^2, \qquad z = x + t,$$

under the constraints that (x, y, z) is primitive (i.e. $\gcd(x, y, z) = 1$) and $x > y$. The analysis characterises the admissible forms of t and shows non-existence for small values such as $t \in \{3, 4, 5, 6, 7\}$.

Appendix G.1 Euclid parametrisation and the two assignments for x

Every primitive Pythagorean triple arises from coprime integers $r > s > 0$ of opposite parity by

$$\begin{aligned} a &= r^2 - s^2, \\ b &= 2rs, \\ c &= r^2 + s^2, \end{aligned}$$

with $\{a, b\}$ the legs and c the hypotenuse. Since $z = x + t$ is the hypotenuse c and x is a leg, there are two (mutually exclusive) primitive assignments:

Case I (odd leg as x)

$$x = r^2 - s^2, \qquad y = 2rs, \qquad z = r^2 + s^2,$$

hence

$$t = z - x = 2s^2.$$

Case II (even leg as x)

$$x = 2rs, \qquad y = r^2 - s^2, \qquad z = r^2 + s^2,$$

hence

$$t = z - x = (r - s)^2.$$

Thus, for primitive triples with $z = x + t$ the parameter t must be of one of the two forms

$$t = 2s^2 \qquad \text{or} \qquad t = (r - s)^2,$$

with $\gcd(r, s) = 1$ and r, s of opposite parity.

Appendix G.2 The inequality $x > y$

Translate $x > y$ into constraints on r, s in each case.

Case I.

With $x = r^2 - s^2$, $y = 2rs$,

$$r^2 - s^2 > 2rs \iff r^2 - 2rs - s^2 > 0 \iff (r - s)^2 > 2s^2,$$

equivalently

$$r > s(1 + \sqrt{2}).$$

Hence in Case I we require $t = 2s^2$ together with $r > s(1 + \sqrt{2})$, $\gcd(r, s) = 1$, and opposite parity.

Case II.

With $x = 2rs$, $y = r^2 - s^2$,

$$2rs > r^2 - s^2 \iff -r^2 + 2rs + s^2 > 0 \iff (r - s)^2 < 2s^2.$$

Hence in Case II we require $t = (r - s)^2$ together with $(r - s)^2 < 2s^2$, $\gcd(r, s) = 1$, and opposite parity.

Appendix G.3 Showing there are no solutions for e.g. $t = 3, 4, 5, 6, 7$

Check each $t \in \{3, 4, 5, 6, 7\}$ against the two forms and the parity/gcd and $x > y$ constraints.

$t = 3$ **:** t odd $\Rightarrow t \neq 2s^2$. 3 is not a perfect square, so $t \neq (r - s)^2$. Hence $t = 3$ is impossible.

$t = 4$ **:** If $t = 2s^2$, then $2s^2 = 4 \Rightarrow s^2 = 2$, impossible. If $t = (r - s)^2$, then $r - s = 2$, so $r = s + 2$, which has the same parity as s and therefore cannot satisfy the opposite-parity requirement for a primitive generator. Hence $t = 4$ is impossible.

$t = 5$ **:** t odd $\Rightarrow t \neq 2s^2$. 5 is not a perfect square, so $t \neq (r - s)^2$. Hence $t = 5$ is impossible.

$t = 6$ **:** If $t = 2s^2$, then $2s^2 = 6 \Rightarrow s^2 = 3$, impossible. 6 is not a perfect square, so $t \neq (r - s)^2$. Hence $t = 6$ is impossible.

$t = 7$ **:** t odd $\Rightarrow t \neq 2s^2$. 7 is not a perfect square, so $t \neq (r - s)^2$. Hence $t = 7$ is impossible.

Appendix G.4 Remarks

The two algebraic forms for t are exhaustive because z is the hypotenuse $r^2 + s^2$ and x must be one of the two Euclid legs. Primitivity forces $\gcd(r, s) = 1$ and opposite parity, and the inequality $x > y$ selects which case is admissible while imposing the inequalities $(r - s)^2 > 2s^2$ (Case I) or $(r - s)^2 < 2s^2$ (Case II). These restrictions rule out $t = 3, 4, 5, 6, 7$ for primitive succession triples with $x > y$.

Appendix G.5 Summary

We analyse integer triples (x, y, z) satisfying

$$x^2 + y^2 = z^2, \qquad z = x + t,$$

under the constraints that (x, y, z) is primitive (i.e. $\gcd(x, y, z) = 1$) and $x > y$. The Euclid parametrisation (with $r > s > 0$, $\gcd(r, s) = 1$, opposite parity) yields the two mutually exclusive primitive assignments:

$$\begin{aligned} &\textbf{Case I:} \quad x = r^2 - s^2, \quad y = 2rs, \quad z = r^2 + s^2, \quad t = z - x = 2s^2, \\ &\textbf{Case II:} \quad x = 2rs, \quad y = r^2 - s^2, \quad z = r^2 + s^2, \quad t = z - x = (r - s)^2. \end{aligned}$$

Appendix G.6 Parity consequences

The parametrisation and primitivity imply the following parity dichotomy.

If t is odd, then x is even and y, z are odd.

Proof: If t is odd it cannot equal $2s^2$, so Case I is excluded. Thus we are in Case II where $t = (r - s)^2$ is an odd perfect square, so $r - s$ is odd. Opposite parity of r, s implies one of r, s is even and the other odd; then

$$x = 2rs \quad \text{is even}, \qquad y = r^2 - s^2 \quad \text{is odd}, \qquad z = r^2 + s^2 \quad \text{is odd}.$$

Primitivity ensures no common factor, so the stated parity pattern holds.

If t is even, then x is odd, y is even, and z is odd.

Proof: If t is even and the triple is primitive, either $t = 2s^2$ (Case I) or $t = (r - s)^2$ with $(r - s)$ even (Case II). In Case II, $r - s$ even forces r, s to have the same parity, contradicting primitivity; hence Case II cannot produce primitive triples when $(r - s)$ is even. Therefore, for primitive triples with even t we must be in Case I with $t = 2s^2$. In Case I, opposite parity of r, s gives

$$x = r^2 - s^2 \quad \text{odd}, \qquad y = 2rs \quad \text{even}, \qquad z = r^2 + s^2 \quad \text{odd}.$$

Thus the stated parity pattern follows.

Appendix G.7 Parity Summary

- An odd t forces the leg assigned to x to be the even leg $2rs$, so x is even and the other two entries are odd.
- An even t in a primitive succession triple must be of the form $2s^2$, forcing the leg assigned to x to be the odd leg $r^2 - s^2$, hence x odd, y even, z odd.

Appendix H Integral Root Theorem

A number of the form $\sqrt[n]{a}$ where n and a are positive integers is either an integer or irrational, in the former case a is the n^{th} power of an integer (Niven [1]). This follows from the Integral Root Theorem where we consider a polynomial:

$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + c_{n-3}x^{n-3} + ... + c_2x^2 + c_1x + c_0 = 0$

where the coefficients are integers. If such an equation has a rational root, it is an integer, moreover, this integer root is a divisor of c_0. Now since $\sqrt[n]{a}$ is a root of $x^n - a = 0$ and if this equation has a rational root, it must be an integer. Furthermore, if $\sqrt[n]{a}$ is an integer, say k , then $a = k^n$. For example, $\sqrt[3]{8}$ is an integer since 8 is a perfect cube 2^3 whereas $\sqrt[3]{9}$ is irrational since 9 is not a perfect cube.

Appendix I Why Ltmax Decreases and Accumulates Near 2

Appendix I.1 Behaviour of $\mathrm{Ltmax}(b)$ Along a Succession-t Family

Figure 14 illustrates the values of $\mathrm{Ltmax}(b)$ for succession–8 and succession–18 triples. Empirically, the values decrease along each family and appear to approach 2. In this section we explain why this behaviour is expected, and how it is consistent with the rigorous finiteness result proved in Appendix J.

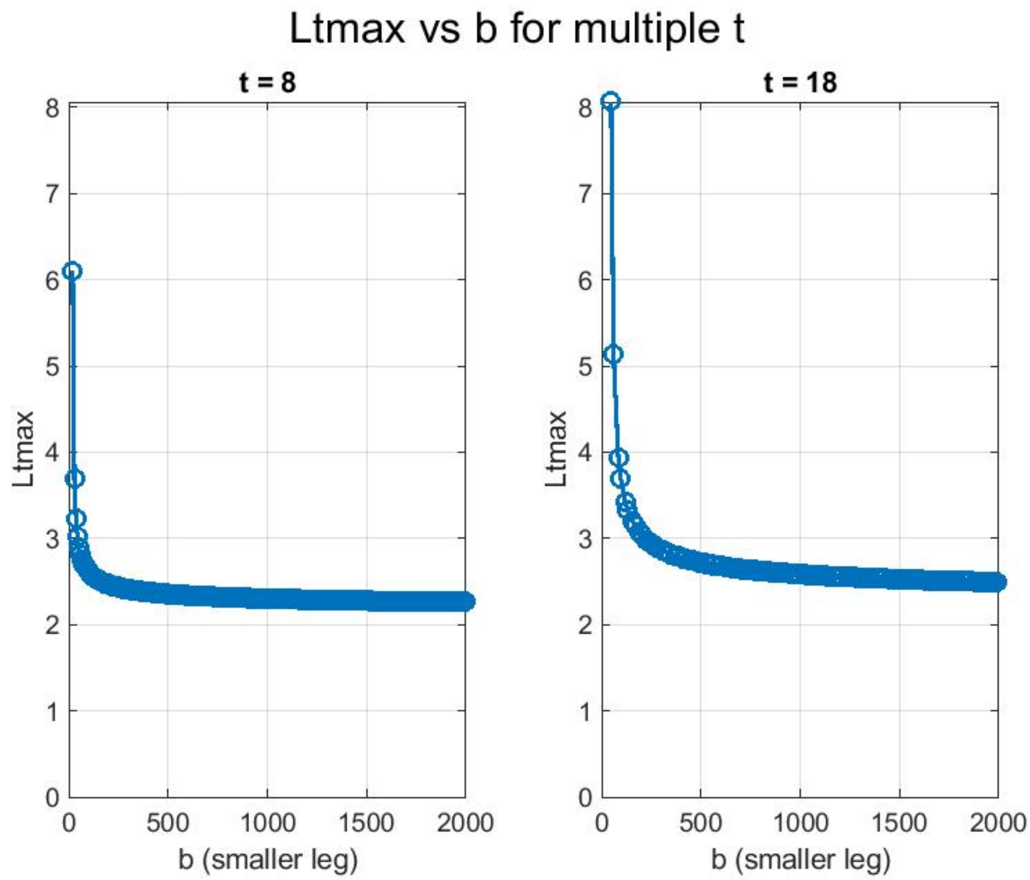


Figure 14: Values of Ltmax(b) for succession–8 and succession–18 triples. Only triples with Ltmax > 3 require explicit checking for integer solutions.

For a fixed succession parameter t, each primitive triple (a, b) in the family determines a unique continuous exponent $m = \text{Ltmax}(b)$ via

$$(a^m + b^m)^{1/m} = a + \delta,$$

where $\delta \in \{1, 2\}$ depends on the parity of t. Equivalently, define

$$f(m) = \ln(a^m + b^m) - m \ln(a + \delta).$$

Then Ltmax(b) is the unique positive root of $f(m) = 0$.

Appendix I.2 Monotonicity of $f(m)$

Differentiating gives

$$f'(m) = \frac{a^m \ln a + b^m \ln b}{a^m + b^m} - \ln(a + \delta).$$

The first term is a weighted average of $\ln a$ and $\ln b$, both strictly less than $\ln(a + \delta)$. Hence

$$f'(m) < 0,$$

so $f(m)$ is strictly decreasing in m, and the root of $f(m) = 0$ is unique for each triple.

Appendix I.3 Asymptotic Interpretation via the Growth of a and b

For a fixed succession parameter t, the Euclid parametrisation shows that s is fixed and $r \to \infty$, giving

$$a = r^2 - s^2 \sim r^2, \qquad b = 2rs \sim 2s\sqrt{a}.$$

Thus

$$b = O(\sqrt{a}), \qquad b^m = O(a^{m/2}).$$

For exponents $m > 2$, the term a^m dominates b^m, and the continuous norm satisfies

$$(a^m + b^m)^{1/m} = a\left(1 + O\left(a^{-(m-2)/2}\right)\right).$$

As the index of the triple increases, the correction term $O(a^{-(m-2)/2})$ becomes negligible. Consequently, the value of m needed to satisfy

$$(a^m + b^m)^{1/m} = a + \delta$$

must move closer to the exponent for which the left-hand side equals the Euclidean norm. Since the Euclidean norm corresponds to $m = 2$, the values Ltmax(b) necessarily accumulate near 2 as $a \to \infty$.

Appendix I.4 Consistency with the Finiteness Theorem

Appendix J establishes that for each fixed t, the condition Ltmax$(a, b) \geq 3$ can hold only for finitely many triples. Beyond the finite index $S_{\max}(t)$, all triples satisfy Ltmax$(a, b) < 3$. The asymptotic analysis above explains the observed behaviour: once a is sufficiently large, the continuous exponent solving $(a^m + b^m)^{1/m} = a + \delta$ must lie in the interval $(2, 3)$, and numerically it approaches 2 from above.

Conclusion. For each fixed succession parameter t, the function Ltmax(b) is well-defined and strictly decreasing in the exponent variable m. Combined with the growth behaviour of a and b along the family, this implies that Ltmax(b) eventually lies below 3 and accumulates near 2 as the triple index increases. This behaviour is consistent with, and helps visualise, the rigorous finiteness of $S_{\max}(t)$.

Appendix J Why Smax(t) Grows with t

Appendix J.1 Variation of Smax with t

Figure 15 shows that Smax(t) increases in a sawtooth manner. For example, for a succession-968 triple e.g. $(2765, 2508, 3733)$ we need to check by hand the first 6429 triples for rationality; after that we then know there are a countably infinite number of succession-968 triples that have no integer solutions.

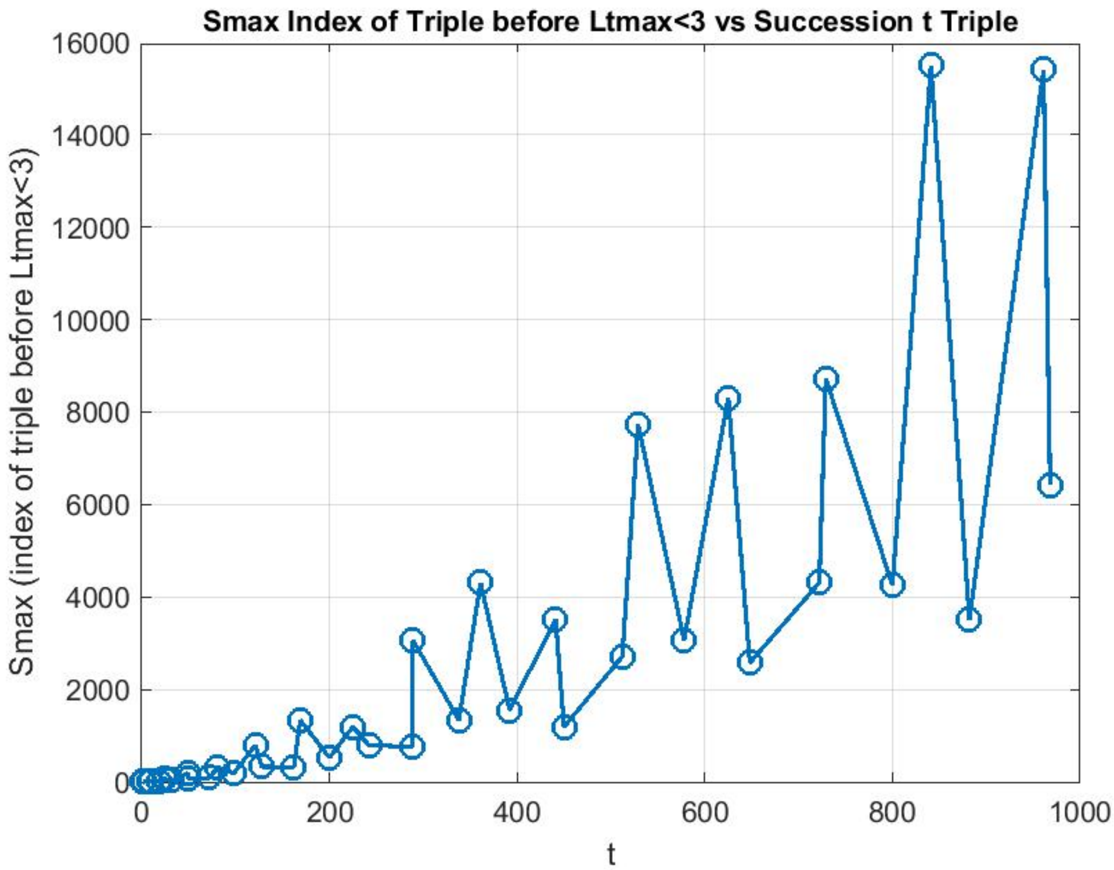


Figure 15: This shows how Smax increases with the t of succession-t triples. Smax is the largest triple index where Ltmax > 3. This indicates the number of triples required to check by hand for integer solution as a function of t for the succession t triple.

Appendix J.2 Finiteness of $S_{\max}(t)$ for Succession-t Families

In this section we prove that for every admissible succession parameter t, only finitely many primitive Pythagorean triples (a, b, c) with $c = a+t$ satisfy $\mathrm{Lt}(a, b) > 3$. Equivalently, the index $S_{\max}(t)$ of the last triple requiring manual checking is finite for each t.

Appendix J.3 Setup and Euclid parametrisation

Recall that every primitive Pythagorean triple can be written in Euclid form

$$a = r^2 - s^2, \qquad b = 2rs, \qquad c = r^2 + s^2,$$

where $r > s > 0$, $\gcd(r, s) = 1$, and r, s have opposite parity.
The succession-t condition $c = a + t$ becomes

$$r^2 + s^2 = (r^2 - s^2) + t,$$

hence

$$t = 2s^2.$$

Thus for any fixed t the parameter s is uniquely determined by

$$s = \sqrt{\tfrac{t}{2}},$$

and r ranges over positive integers satisfying the usual coprimality and parity conditions with this fixed s.
In particular, for fixed t we may regard s as a constant and let $r \to \infty$. The corresponding legs and hypotenuse are

$$a = r^2 - s^2, \qquad b = 2rs, \qquad c = r^2 + s^2 = a + t.$$

Appendix J.4 Asymptotic growth of a and b for fixed t

Fix an admissible succession parameter t, and let (a, b, c) be the corresponding primitive triples with $c = a + t$. Then, as $r \to \infty$,

$$a \sim r^2, \qquad b \sim 2sr,$$

and in particular

$$b = O(\sqrt{a}) \quad \text{and} \quad b^3 = O(a^{3/2}).$$

Since s is fixed and $r \to \infty$, we have

$$a = r^2 - s^2 = r^2\Big(1 - \frac{s^2}{r^2}\Big) \sim r^2,$$

and

$$b = 2rs \sim 2sr.$$

Solving asymptotically for r from $a \sim r^2$ gives $r \sim \sqrt{a}$, hence

$$b \sim 2s\, r \sim 2s\, \sqrt{a},$$

which shows $b = O(\sqrt{a})$. Cubing this relation yields

$$b^3 = O(a^{3/2}).$$

Appendix J.5 The threshold condition $Lt(a, b) > 3$

By definition, $\mathrm{Lt}(a, b)$ is the unique real m such that

$$(a^m + b^m)^{1/m} = a + \delta,$$

where $\delta \in \{1, 2\}$ is determined by the parity of t (as in the main text). In particular,

$$\mathrm{Lt}(a, b) > 3 \quad \Longleftrightarrow \quad a^3 + b^3 > (a + \delta)^3,$$

since the mth root is strictly increasing in m for fixed positive a, b.
Expanding the right–hand side gives

$$(a + \delta)^3 = a^3 + 3\delta a^2 + 3\delta^2 a + \delta^3,$$

so that

$$\mathrm{Lt}(a, b) > 3 \quad \Longleftrightarrow \quad b^3 > 3\delta a^2 + 3\delta^2 a + \delta^3.$$

In particular, a necessary condition for $\mathrm{Lt}(a, b) > 3$ is

$$b^3 > 3\delta a^2.$$

For fixed t (hence fixed δ and s), the inequality

$$b^3 > 3\delta a^2$$

fails for all sufficiently large primitive triples in the succession-t family. Equivalently, there exists $A(t)$ such that

$$a > A(t) \quad \Longrightarrow \quad b^3 \le 3\delta a^2.$$

By Lemma Appendix J.4, we have $b^3 = O(a^{3/2})$ as $a \to \infty$ for fixed t. More concretely, there exists a constant $K(t) > 0$ such that

$$b^3 \le K(t)\, a^{3/2}$$

for all sufficiently large a in the succession-t family. For such a we therefore have

$$\frac{b^3}{a^2} \le K(t)\, a^{-1/2} \longrightarrow 0 \quad \text{as } a \to \infty.$$

Hence we may choose $A(t)$ so large that

$$K(t)\, a^{-1/2} \le 3\delta \quad \text{whenever } a > A(t),$$

which implies

$$b^3 \le K(t)\, a^{3/2} \le 3\delta a^2.$$

Thus $b^3 > 3\delta a^2$ can hold only for finitely many values of a (and hence finitely many triples) in the succession-t family.

Appendix J.6 Main theorem: Finiteness of $S_{\max}(t)$

For every admissible succession parameter t, there are only finitely many primitive Pythagorean triples (a, b, c) with $c = a + t$ satisfying $\mathrm{Lt}(a, b) > 3$. In particular, the index

$$S_{\max}(t) = \max\{\, k : \mathrm{Lt}(a_k, b_k) \ge 3 \,\}$$

is finite for each t.
As observed above, $\mathrm{Lt}(a, b) > 3$ implies

$$b^3 > 3\delta a^2 + 3\delta^2 a + \delta^3 \quad > \quad 3\delta a^2.$$

By Lemma Appendix J.5, the inequality $b^3 > 3\delta a^2$ can hold only for finitely many triples in the succession-t family. Hence $\mathrm{Lt}(a, b) > 3$ can also hold only for finitely many such triples.
Since the primitive succession-t triples can be indexed in order of increasing a (or any other fixed ordering), there exists a largest index $S_{\max}(t)$ for which $\mathrm{Lt}(a_k, b_k) \ge 3$. For all triples beyond this index, we have $\mathrm{Lt}(a_k, b_k) < 3$, and no manual checking is required.

Appendix J.7 Interpretation

Theorem Appendix J.6 shows that for any fixed succession parameter t, only finitely many triples require explicit verification. Beyond the finite index $S_{\max}(t)$, all triples in the succession-t family have continuous norm exponent $\mathrm{Lt}(a, b) < 3$, and thus cannot yield integer solutions to $x^n + y^n = z^n$ for $n > 2$ along the corresponding rational directions.

Appendix K Possible Extensions Beyond Succession-t Families

The finiteness theorem proved in Appendix J establishes that for each fixed succession parameter t, only finitely many primitive Pythagorean triples in the corresponding family satisfy $\mathrm{Lt}(a, b) \ge 3$. This result relies crucially on the rigidity introduced by fixing the hypotenuse gap $c - a = t$, which forces the Euclid parameter s to remain constant while $r \to \infty$. Consequently,

$$a = r^2 - s^2 \sim r^2, \qquad b = 2rs = O(\sqrt{a}),$$

and the inequality $b^3 > 3\delta a^2$ required for $\mathrm{Lt}(a, b) > 3$ eventually becomes impossible. It is natural to ask whether analogous finiteness or asymptotic phenomena hold for *all* primitive triples, without fixing t. In this section we outline several directions in which the present methods may be extended.

Appendix K.1 Classification by Aspect Ratio b/a

For general primitive triples, the Euclid parametrisation

$$a = r^2 - s^2, \qquad b = 2rs$$

allows both parameters r and s to vary. The ratio

$$\lambda = \frac{b}{a} = \frac{2rs}{r^2 - s^2}$$

serves as a natural "shape parameter" describing the geometry of the triple. Succession-t families correspond to the extreme regime $\lambda \to 0$, where the triangles become increasingly thin. A possible extension is to fix $\lambda \in (0, 1)$ and study triples for which $b/a \approx \lambda$. One may then ask whether the continuous exponent $\mathrm{Lt}(a, b)$ converges to a limit $m(\lambda)$ as $a \to \infty$, and how $m(\lambda)$ varies with λ. The present work effectively determines the limiting behaviour in the case $\lambda \to 0$.

Appendix K.2 Asymptotics in Multiple Growth Regimes

The finiteness of $S_{\max}(t)$ relies on the growth law $b = O(\sqrt{a})$ that arises when s is fixed. For general triples, however, the parameter ratio s/r may tend to any value in $(0, 1)$, and the growth of b may range from $O(\sqrt{a})$ up to $\Theta(a)$. Extending the analysis of $\mathrm{Lt}(a, b)$ to these broader regimes would require a more delicate study of the function

$$(a^m + b^m)^{1/m}$$

as both a and b grow in a coupled manner. One possible goal is to identify conditions on the growth of b relative to a under which $\mathrm{Lt}(a, b)$ remains below 3 for all sufficiently large triples.

Appendix K.3 Density of Triples with Large Lt

Another natural question is whether triples with large continuous exponent are rare in a global sense. For example, one may ask whether the set of primitive triples satisfying $\mathrm{Lt}(a, b) \geq 3$ has density zero when triples are ordered by increasing hypotenuse or by increasing Euclid parameter r. The methods developed for succession-t families suggest that large values of $\mathrm{Lt}(a, b)$ require the leg b to grow unusually quickly relative to a, which may be a statistically rare phenomenon.

Appendix K.4 Geometric Interpretation on Fermat Curves

The geometric viewpoint developed in this paper - interpreting each triple as a rational direction intersecting the Fermat curve $x^p + y^p = 1$ - extends naturally to all primitive triples. A possible line of investigation is to classify the rational directions that yield irrational intersections for all $p > 2$, and to determine whether such directions form a set of full measure. The succession-t families provide explicit infinite subfamilies with this property; extending this to a global classification would be a significant advance.

Appendix K.5 Summary

The finiteness of $S_{\max}(t)$ demonstrates that fixing the hypotenuse gap imposes strong asymptotic constraints on the geometry of the corresponding triples. Extending these results to all primitive triples would require new ideas, particularly in analysing the behaviour of $\mathrm{Lt}(a, b)$ across different growth regimes for a and b. The directions outlined above represent natural avenues for future research and illustrate how the geometric and analytic framework developed in this paper may be broadened beyond the succession-t setting.

References

[1] Ivan Niven, *Numbers Rational and Irrational*, The Mathematical Association of America, 1961, p. 61.